Local convergence for the curve tracing of the homotopy method

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Abstract. The local convergence of a Newton-method for the tracing of an implicitly defined smooth curve is analyzed. The domain of attraction is shown to be larger than in [6]. Moreover finer error bounds on the distances involved are obtained and quadratic instead of geometrical order of convergence is established. A numerical example is also provided where our results compare favourably with the corresponding ones in [6].

Keywords and phrases. Curve tracing, homotopy method, domain of attraction, radius of convergence, Newton-Kantorovich theorem/hypothesis, smooth curve, Moore-Penrose generalized inverse.


Resumen. Se analiza la convergencia local de un método de Newton para el trazado de una curva suave definida implícitamente. Se muestra que el dominio de atracción es más grande que en [6]. Además se obtienen errores más finos para las cotas de las distancias involucradas y se establece orden cuadrático en lugar de lineal para la convergencia. Se da un ejemplo numérico donde nuestro resultado se compara favorablemente con los resultados correspondientes en [6].

1. Introduction

We are concerned with the following problem: Suppose that a smooth curve $\Gamma \subset \mathbb{R}^{n+1}$ is implicitly defined by

$$F(x, t) = 0,$$

where $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a $C^2$ function. We intend to numerically trace curve $\Gamma$ from the point $(x_0, t_0)$ to the point $(x^*, t^*)$. We assume the $n \times (n + 1)$ Jacobian matrix $DF(x, t)$ has full rank at every point in $\Gamma$. A survey of such techniques can be found in [1], [8] and the references there.

We will use the following algorithmic form:
Let \( y_i = (x_i, t_i) \in \mathbb{R}^{n+1} \) be an approximation for \( \Gamma \). Use the predictor
\[
z_0 = y_i + h_i \tau_i
\] (1.2)
for the next approximating point, where \( h_i \) is an appropriate step length and \( \tau_i \) is the tangent vector of \( \Gamma \) at \( y_i \);

(b) Starting from \( z_0 \), take a sequence of Newton iterations by requiring \( z_k \) to lie on the hyperplane normal to a certain vector (usually the tangent vector \( \tau_i \));

(c) Set \( y_{i+1} = z \) where \( z \) is the point of convergence for the sequence \( \{ z_k \} \).

We need some preliminaries:
A point \((x, t)\) in \( \mathbb{R}^{n+1} \) will be denoted by \( y \). Let \( \sigma \) be the arc length, along the curve \( \Gamma \), then an initial value problem is implicitly defined by
\[
DF(y) \cdot \dot{y} = 0; \quad y(0) = y_0, \quad (1.3)
\]
where \( \cdot = \frac{d}{d\sigma} \). It is known that vector field \( \dot{y} \) is locally Lipschitzian [8].

We assume \( DF(y) \) is full rank along the solution curve, then equation
\[
DF(y) y' = -F(y) \quad (1.4)
\]
can be reduced to
\[
y' = -DF^+(y) F(y) \quad (1.5)
\]
where \( DF^+(y) = DF^T(y) [DF(y) DF^T(y)]^{-1} \) is the Moore-Penrose generalized inverse of \( DF(y) \). By the result
\[
\text{rang} (DF^+) = \text{rang} (DF^T) = \ker (DF) \perp \quad (1.6)
\]
and equation
\[
F(y(\tau)) = e^{-\tau} F(y(0)) \quad (1.7)
\]
we conclude a solution \( y(\tau) \) of (1.5) is such that the magnitude of \( F(y) \) is reduced and also remains perpendicular to the 1-dimensional kernel space of \( F(y) \).

Consider the Euler step of (1.5). This corresponds to the Newton method in the form
\[
y_{k+1} = y_k - DF^+(y_k) F(y_k). \quad (1.8)
\]
In the next section we analyze the local convergence of method (1.8).

We state a result whose proof can be found in [6, p. 327]:

**Theorem 1.1.** Let \( F : D \subseteq \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n \) be a \( C^2 \) function such that
\[
\|DF(x) - DF(y)\| \leq \ell \|x - y\|, \quad \text{for all } x, y \in D. \quad (1.9)
\]
Suppose that \( F(x^*) \) and \( DF(x^*) \) is full rank. Let \( \delta \in \left(0, \frac{3-\sqrt{5}}{2}\right) \) and define
\[
M = \min \left\{ \frac{2}{3\|DF^+(x^*)\| \ell}, \frac{\text{dist}(x^*, \partial D)}{\delta} \right\}. \quad (1.10)
\]
If \( r \in (0, \delta M = r_0) \) is such that for every \( x \in U(x^*, r) = \{ x \in \mathbb{R}^{n+1} : \|x - x^*\| \leq r \} \) we have
\[
\|F(x)\| \leq \frac{\delta \ell M^2}{2},
\] (1.11)
then for any \( x_0 \in U(x^*, r) \subseteq D \), method (1.8) is well defined and converges geometrically to a point in \( \Gamma \cap U(x^*, M) \).

**Remark 1.1.** Under the hypotheses of Theorem 1.1 method (1.8) converges only geometrically and condition (1.1) should hold. To do so we first introduce the center Lipschitz condition
\[
\|DF(x) - DF(x^*)\| \leq \ell_0 \|x - x^*\|, \quad \text{for all } x \in D.
\] (1.12)

We note that in general
\[
\ell_0 \leq \ell
\] (1.13)
holds and \( \frac{\ell}{\ell_0} \) can be arbitrarily large. In practice the computation of \( \ell_0 \) requires that of \( \ell_0 \).

Then we can show the following improvement over Theorem 1.1.

**Theorem 1.2.** Suppose hypotheses of Theorem 1.1 and (1.12) hold but \( M \) is defined as
\[
M_0 = \min \left\{ \frac{2}{(2\ell_0 + \ell) \|DF^+(x^*)\|}, \ dist(x^*, \partial D) \right\},
\] (1.14)
then the conclusions of Theorem 1.1 hold with \( M_0 \) replacing \( M \).

**Proof.** For any \( x \in U(x^*, M_0) \), we get using Lemma 3.1 in [6, p. 326] and (1.12):
\[
\|DF(x) - DF(x^*)\| \|DF^+(x^*)\| \leq \ell_0 \|x - x^*\| \|DF^+(x^*)\| < \frac{2}{3} < 1.
\] (1.15)
The rest of the proof follows exactly as in Theorem 1 in [6, p. 326] (with \( M_0 \) replacing \( M \)). That completes the proof of the theorem. \( \Box \)

**Remark 1.2.** If equality holds in (1.13) then Theorem 1.2 reduces to Theorem 1.1. Otherwise
\[
M < M_0
\] (1.16)
holds and the bounds on the distances \( \|y_{n+1} - y_n\|, \|y_{n+1} - x^*\| \) \( (n \geq 0) \) are finer in Theorem 1.2. This improvement allows a wider choice of initial guesses \( x_0 \). Such an observation is important in computational mathematics. By comparing (1.10) and (1.14) we see that \( M_0 \) can be (at most) three times larger than \( M \) (if \( \ell_0 = \ell \)).

In order to show that it is possible to achieve quadratic convergence and drop strong condition (1.11) we use a modification of our Theorem 2 in [3] (where we have replaced \( F'(x)^{-1} \) by \( DF(x)^+ \) and use Lemma 3.1 in [6] instead of Banach Lemma on invertible operators in the proof of Theorem 2 in [3]) to obtain the proof of Theorem 1.3 that follows:
Theorem 1.3. Assume conditions of Theorem 1.2 hold excluding (1.11). If
\[ U_1(x^*, r_1) \subseteq D, \]
where
\[ r_1 = \frac{1}{\ell_0 \|DF(x^*)^+\|}, \quad (1.18) \]
then for all \( x_0 \in U_2(x^*, r_2) \), where
\[ r_2 = \frac{2 + \gamma - \sqrt{\gamma^2 + 2\gamma}}{(2 + \gamma)\ell_0 \|DF(x^*)^+\|}, \quad \text{for } \gamma \geq 2, \ell = \frac{\gamma}{2} \ell_0, \quad (1.19) \]
the following hold:

Newton-Kantorovich hypothesis
\[ h = 2\ell \|DF(x_0)^+\| \|DF(x_0)^+ F(x_0)\| \leq 1 \quad (1.20) \]
holds as strict inequality, and consequently the Newton-Kantorovich theorem guarantees method (1.8) is well-defined and converges quadratically to a point in \( \Gamma \cap U(x^*, r_1) \).

Remark 1.3. Even if equality holds in (1.13) we can set \( \gamma = 2 \) and \( r_2 \) can be written as
\[ r_2 = \frac{2 - \sqrt{2}}{2\ell_0 \|DF(x^*)^+\|}, \quad (1.21) \]
which is larger than \( r_0 \) since
\[ \delta < \frac{2 - \sqrt{2}}{2}. \quad (1.22) \]
If strict inequality holds in (1.13) then \( r_2 \) is enlarged even further (see also Example 1.4 as follows).

Convergence radius \( r_2 \) can be extended even further by using Theorem 3 in [3] based on an even weaker hypotheses than (1.20) found by us in Section 1.2:
\[ h_0 = (\ell + \ell_0) \|DF(x_0)^+\| \|DF(x_0)^+ F(x_0)\| \leq 1. \quad (1.23) \]
However we do not pursue this here, leaving it for the motivated reader.

Instead we provide an example where strict inequality holds in (1.13).

Example 1.4. Let \( D = U(0, 1) \) and define function \( F \) on the real line by
\[ F(x) = e^x - 1. \quad (1.24) \]
For simplicity we take $x_0 = x^*$. We obtain

$$
\ell = e, \\
\ell_0 = e - 1, \\
\|DF(x^*)^+\| = 1, \\
\gamma = 3.163953415, \\
\delta = .381966011, \\
M = .245252961, \\
M_0 = .324947231, \\
r_0 = \delta M = .093678295, \\
r_1 = .581976707, \\
r_2 = .126433594.
$$

Therefore we conclude

$$M < M_0 < r_1$$

and

$$r_0 < \tilde{r}_0 < r_2,$$

which demonstrate the superiority of our results over the ones in [6].

References


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