On the homeotopy group of the non orientable surface of genus three

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ABSTRACT. In this note we prove that, if \( N_3 = P \# P \# P \), where \( P := \mathbb{R}P^2 \), then the canonical homomorphism from \( \text{Diff}(N_3) \) onto the homeotopy group \( \text{Mod}(N_3) \) has a section. To do this we first prove that \( \text{Mod}(N_3) = \text{GL}(2, \mathbb{Z}) \).

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RESUMEN. En esta nota probamos que, si \( N_3 = P \# P \# P \), donde \( P := \mathbb{R}P^2 \), entonces el homomorfismo canónico de \( \text{Diff}(N_3) \) sobre el grupo de homeotopía \( \text{Mod}(N_3) \) tiene una sección. Para hacer esto, primero probamos que \( \text{Mod}(N_3) = \text{GL}(2, \mathbb{Z}) \).

1. Introduction

If \( M \) is a closed smooth surface we denote by \( \text{Mod}(M) \) the quotient group \( \text{Diff}(M)/\text{Diff}_0(M) \) where \( \text{Diff}(M) \) is the group of all diffeomorphisms from \( M \) to \( M \) and \( \text{Diff}_0(M) \) is the normal subgroup of diffeomorphisms isotopic to the identity. We call it the homeotopy group or the extended mapping class group of \( M \).

S. Morita [9], [10] has shown that, if \( M_g \) is the closed genus \( g \) orientable surface, then the canonical epimorphism

\[ \text{Diff}(M_g) \to \text{Mod}(M_g) \]

from the group of diffeomorphisms of \( M_g \) onto its extended mapping class group admits no section provided that \( g \geq 18 \).

When \( g \leq 1 \) it is easy to show that the homomorphism does have a splitting: If \( g = 0 \) then \( \text{Mod}(M_0) = \mathbb{Z}_2 \); a section is defined by sending the non trivial element of \( \text{Mod}(M_0) \) to the antipodal map of \( S^2 \). Also, for genus one \( M_1 = \)
Lemma 2.1. \( \mathbb{R}^2/\mathbb{Z}^2 \) and \( \text{Mod}(M_1) = GL(2, \mathbb{Z}) \) (cf. [11, p. 26]). The standard linear action of \( GL(2, \mathbb{Z}) \) on \( (\mathbb{R}^2, \mathbb{Z}^2) \) defines a splitting of \( \text{Diff}(M_1) \to \text{Mod}(M_1) \).

If \( N_k \) is the genus \( k \) non-orientable surface (the connected sum of \( k \) copies of \( P \)) then

\[
\text{Diff}(N_k) \to \text{Mod}(N_k),
\]

has a section if \( k \leq 2 \).

For, if \( k = 1 \) then \( \text{Mod}(P) = 1 \) (see [4]) and trivially a section exists. If \( k = 2 \) and we think of \( N_2 \) as \( S^1 \times S^1 \) with identifications \( (z, w) \sim (-z, \bar{w}) \), then \( \text{Mod}(N_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) and the image of a section is \( \{ f_{\epsilon_1, \epsilon_2} : |\epsilon_1| = |\epsilon_2| = 1 \} \) where \( f_{\epsilon_1, \epsilon_2}(z, w) = (z^{\epsilon_1}, w^{\epsilon_2}) \) (see [6], [12]).

Here we will prove that

\[
\text{Diff}(N_k) \to \text{Mod}(N_k),
\]

also has a section if \( k = 3 \).

2. Proofs

First, we will show that \( \text{Mod}(N_3) = GL(2, \mathbb{Z}) \).

In [2], using [3], a presentation of \( \text{Mod}(N_3) \) is given and one can see that this presentation defines \( GL(2, \mathbb{Z}) \), (see [7]).

However we feel that this result is not well known. In here we will give a proof of the fact that \( \text{Mod}(N_3) = \text{Mod}(M_1)(= GL(2, \mathbb{Z})) \) using simple methods in algebraic topology.

We will work in the smooth category.

Let \( T_0 \) be a torus minus the interior of a 2–disk \( D \). An arc \( \alpha \) properly embedded in \( T_0 \) is trivial if there is a 2–disk in \( T_0 \) whose boundary is the union of \( \alpha \) and an arc in \( \partial T_0 \). This is equivalent to the condition that \( \alpha \) represent the trivial element of \( H_1(T_0, \partial T_0; \mathbb{Z}_2) \). In the following lemma \( \cup_{i=1}^{n} \alpha_i/\varphi \) will denote the quotient space of the union of arcs \( \cup_{i=1}^{n} \alpha_i \) obtained by identifying \( x \in \partial (\cup_{i=1}^{n} \alpha_i) \) with \( \varphi(x) \).

Lemma 2.1. Let \( T_0 \) be the torus minus the interior of a 2–disk. Let \( \varphi : \partial T_0 \to \partial T_0 \) be a fixed point free involution. Let \( \alpha_1, \ldots, \alpha_n \), with \( n \) odd, be disjoint arcs properly embedded in \( T_0 \) such that \( \varphi \partial (\cup_{i=1}^{n} \alpha_i) = \partial (\cup_{i=1}^{n} \alpha_i) \), \( \cup_{i=1}^{n} \alpha_i/\varphi \) is connected and \( \sum_{i=1}^{n} |\alpha_i| = 0 \in H_1(T_0, \partial T_0; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). Then at least one \( \alpha_i \) is trivial.

Proof. Let \( a, b, c \) be the nontrivial elements of \( H_1(T_0, \partial T_0; \mathbb{Z}_2) \). Let \( a_1, \ldots, a_p \) be the arcs of \( \{ \alpha_1, \ldots, \alpha_n \} \) which represent \( a \). Let \( b_1, \ldots, b_q \) those which represent \( b \) and \( c_1, \ldots, c_r \) those which represent \( c \).

Assume no \( \alpha_i \) is trivial, that is \( |\alpha_i| \neq 0 \) for all \( i \). Then \( 0 = \sum_{i=0}^{n} |\alpha_i| = pa + bq + rc \) in \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) and \( p + q + r = n \), an odd number. If one of the numbers \( p, q, r \) is even then the other two must also be even; but this contradicts the fact that \( n \) is odd. Therefore \( p, q, r \) are all odd.

Notice that for any \( i \) and any \( j \) the 0–spheres \( \partial \alpha_i \) and \( \partial b_j \) are linked in \( \partial T_0 \) (meaning that both components of \( \partial T_0 - \partial \alpha_i \) contain one point of \( \partial b_j \)).
Similarly $\partial b_i$ and $\partial c_k$ are linked, and $\partial c_k$ and $\partial a_i$ are linked, for any values of $i, j, k$. Also $\partial a_i$ and $\partial a_j$ are not linked $\partial b_i$ and $\partial b_j$ are not linked, $\partial c_i$ and $\partial c_k$ are not linked if $i \neq j$.

**Figure 1**

This implies that after renumbering the $a$’s, $b$’s and $c$’s the arrangement of the points of $\cup_{i=1}^{p} \partial a_i$ in $\partial T_0$ is: $a_1^+, a_2^+, \ldots, a_p^+, b_1^+, \ldots, b_q^+, c_1^+, \ldots, c_k^+, a_{p-1}^-, a_{p-2}^-, \ldots, a_2^-, a_1^-, b_q^-, \ldots, b_1^-, c_k^-, \ldots, c_1^-$ as shown in figure 1; here $\partial a_i = \{a_i^+, a_i^−\}$, $\partial b_j = \{b_j^+, b_j^−\}$ and $\partial c_k = \{c_k^+, c_k^−\}$.

But then the number of components of $\cup a_i / \varphi$ is $\frac{p+1}{2} + \frac{q+1}{2} + \frac{r+1}{2} > 1$ (think of $\varphi$ as the antipodal involution), contradicting that $\cup a_i / \varphi$ is connected. Hence at least one $a_i$ is trivial. $\Box$

We write $N = N_3$ henceforth.

**Proposition 2.1.** Let $\mu$ and $\alpha$ be simple closed curves in $N$ representing the element of order 2 in $H_1(N; \mathbb{Z})$. Then $\alpha$ is isotopic to $\mu$.

**Proof.** Write $N = T_0 \cup P_0$, the union of a punctured torus $T_0$ and a Möbius band $P_0$, with $T_0 \cap P_0 = \partial T_0 = \partial P_0$. We think of $P_0$ as an $I$–bundle over the circle and denote by $\varphi: \partial T_0 \to \partial T_0$ the fixedpoint free involution that interchanges the boundary points of each fiber.

We may assume that $\mu$ is the image of a section of this bundle. We may also assume that $\alpha$ intersects $\partial T_0$ minimally, that is, $|a \cap \partial T_0| \geq |a \cap \partial T_0|$ for any curve $a$’ ambient isotopic to $\alpha$. We claim that $|\alpha \cap \partial T_0| = 0$.

Suppose $|\alpha \cap \partial T_0| > 0$. Then we can assume that $\alpha \cap P_0$ consists of $n$ $I$–fibers $f_1, \ldots, f_n$ and $\alpha \cap T_0$ is the union of $n$ disjoint arcs $\alpha_1, \ldots, \alpha_n$ properly embedded in $T_0$. As $H_1(N) = H_1(P_0, \partial P_0) \oplus H_1(T_0, \partial T_0) = \mathbb{Z}_2 \oplus \mathbb{Z}^2$ and as $\alpha$ represents the element of order two, then we must have that $\sum \{\alpha_i \} \neq 0$ in $H_1(T_0, \partial T_0; \mathbb{Z}_2)$ (that is, $n$ must be odd) and $\sum \{\alpha_i \} = 0$ in $H_1(T_0, \partial T_0; \mathbb{Z}_2)$. By Lemma 2.1, at least one $\alpha_i$ must be trivial and so we can isotope $\alpha_i$ to reduce the number of components of its intersection with $\partial T_0$. This contradicts our minimality assumption. Hence $|\alpha \cap \partial T_0| = 0$ and, since $\alpha$ is not trivial, it is isotopic to $\mu$. $\Box$

**Proposition 2.2.** Let $N = T_0 \cup P_0$ with $T_0 \cap P_0 = \partial T_0 = \partial P_0$. Then any diffeomorphism $h$ of $N$ is isotopic to one leaving $T_0$ and $P_0$ invariant.

**Proof.** Let $\mu$ be the image of a section of $P_0$. By Proposition 2.1, $h\mu$ is ambient isotopic to $\mu$ so we may assume that $h$ leaves $\mu$ invariant. But then we can also assume that it leaves its tubular neighborhood $P_0$ invariant. $\Box$
Theorem 2.3. The natural homomorphism
\[ \psi : \operatorname{Mod}(N) \to \operatorname{Aut}(H_1(N)/\operatorname{Torsion}(H_1(N)) \cong GL(2, \mathbb{Z})) , \]
is an isomorphism.

Proof. Again write \( N = T_0 \cup P_0 \) and \( T = T_0 \cup D \). Any automorphism of \( H_1(T) \) is induced by a diffeomorphism of \( T \) which can be isotoped so that the 2−disk \( D \) is invariant. Hence any automorphism of \( H_1(T_0, \partial T_0) \) is induced by a diffeomorphism of \( T_0 \). Since any diffeomorphism of \( \partial P_0 \) can be extended to a diffeomorphism of \( P_0 \) (a nice exercise), it follows that any automorphism of \( H_1(N)/\operatorname{Torsion}(H_1(N)) \) is induced by a diffeomorphism of \( N \). Thus \( \psi \) is a monomorphism. □✓

Theorem 2.4. The natural homomorphism \( \operatorname{Diff}(N) \to \operatorname{Mod}(N) \) has a section.

Proof. Let \( T = \mathbb{R}^2/\mathbb{Z}^2 \). Consider the blow up \( B(T) \) of \( T \) at the identity element \( e \) of \( T \). Recall \( B(T) = (T - \{e\}) \cup P^1 \) where \( P^1 \) is the space of one-dimensional vector subspaces of \( \mathbb{R}^2 \). The blow up \( B(T) \) is diffeomorphic to \( N \).

If \( f \) is a linear automorphism of \( \mathbb{R}^2 \) with \( f(\mathbb{Z}^2) = \mathbb{Z}^2 \), it induces a diffeomorphism of \( T - \{e\} \), a diffeomorphism of \( P^1 \) and a diffeomorphism of \( B(T) \) (cf.[1, Lemma 2.1]). Thus the standard linear action of \( GL(2, \mathbb{Z}) \) on \( T \) induces an action of \( GL(2, \mathbb{Z}) \) on \( B(T) \).

Hence we have a homomorphism
\[ GL(2, \mathbb{Z}) \to \operatorname{Diff}(B(T)) , \]
which composed with
\[ \operatorname{Diff}(B(T)) \to \operatorname{Mod}(B(T)) \cong \operatorname{Aut}(H_1(N)/\operatorname{Torsion}(H_1(N))) \]
is an isomorphism. □✓

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References


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