Some results on the geometry of full flag manifolds and harmonic maps

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Abstract. In this note we study, for \( n = 5, 6, 7 \), the geometry of the full flag manifolds, \( F(n) = U(n)/O(n) \). By using tournaments we characterize all of the (1,2)-symplectic invariant metrics on \( F(n) \), for \( n = 5, 6, 7 \), corresponding to different classes of non-integrable invariant almost complex structure.

Keywords and phrases. Flag manifolds, (1,2)-symplectic metrics, harmonic maps, Hermitian geometry.

2000 Mathematics Subject Classification. Primary 53C15. Secondary 53C55, 14M15, 58E20, 05C20.

1. Introduction

Eells and Sampson [ES], proved that if \( \phi: M \to N \) is a holomorphic map between Kähler manifolds then \( \phi \) is harmonic. This result was generalized by Lichnerowicz (see [L] or [Sa]) as follows: Let \((M, g, J_1)\) and \((N, h, J_2)\) be almost Hermitian manifolds with \( M \) cosymplectic and \( N \) (1,2)-symplectic. Then any \( \pm \) holomorphic map \( \phi: (M, J_1) \to (N, J_2) \) is harmonic.

We are interested to study harmonic maps, \( \phi: M^2 \to F(n) \), from a closed Riemannian surface \( M^2 \) to a full flag manifold \( F(n) \). Then by the Lichnerowicz theorem, we must study (1,2)-symplectic metrics on \( F(n) \), because a Riemannian surface is a Kähler manifold and a Kähler manifold is a cosymplectic manifold (see [Sa] or [GH]).

*Supported by CAPES (Brazil) and COLCIENCIAS (Colombia).
The study of invariant metrics on $F(n)$ involves almost complex structures on $F(n)$. Borel and Hirzebruch [BH], proved that there are $2^{(\frac{n}{2})}$ $U(n)$-invariant almost complex structures on $F(n)$. This number is the same number of tournaments with $n$ players or nodes. A tournament is a digraph in which any two nodes are joined by exactly one oriented edge (see [M] or [BS]). There is a natural identification between almost complex structures on $F(n)$ and tournaments with $n$ players, see [MN3] or [BS].

The tournaments can be classified in isomorphism classes. In that classification, one of these classes corresponds to the integrable structures and the another ones correspond to non-integrable structures. Burstall and Salamon [BS], proved that a almost complex structure $J$ on $F(n)$ is integrable if and only if the associated tournament to $J$ is isomorphic to the canonical tournament (the canonical tournament with $n$ players, $\{1, 2, \ldots, n\}$, is defined by $i \to j$ if and only if $i < j$). In that paper the identification between almost complex structures and tournaments plays a very important role.

Borel [Bo], proved that exits a $(n-1)$-dimensional family of invariant Kähler metrics on $F(n)$ for each invariant complex structure on $F(n)$. Eells and Salamon [ESa], proved that any parabolic structure on $F(n)$ admits a (1,2)-symplectic metric. Mo and Negreiros [MN2], showed explicitly that there is a $n$-dimensional family of invariant (1,2)-symplectic metrics for each parabolic structure on $F(n)$, the identification between almost complex structures and tournaments is strongly used in that paper.

Mo and Negreiros ([MN1], [MN2]) studied the geometry of $F(3)$ and $F(4)$. In this paper we study the $F(5)$, $F(6)$ and $F(7)$ cases. We obtain the following families of (1,2)-symplectic invariant metrics, different to the Kähler and parabolic: On $F(5)$, two 5-parametric families; on $F(6)$, four 6-parametric families, two of them generalizing the two families on $F(5)$ and, on $F(7)$ we obtain eight 7-parametric families, four of them generalizing the four ones on $F(6)$.

These metrics are used to produce new examples of harmonic maps $\phi: M^2 \to F(n)$, applying the result of Lichnerowicz mentioned above.

These notes are part of the author's Doctoral Thesis [P]. I wish to thank my advisor Professor Caio Negreiros for his right advise. I would like to thank Professor Xiaohuan Mo for his helpful comments and discussions on this work.

2. Preliminaries

A full flag manifold is defined by

$$F(n) = \{(L_1, \ldots, L_n) : L_i \text{ is a subspace of } \mathbb{C}^n, \dim_{\mathbb{C}} L_i = 1, \ L_i \perp L_j\}.$$
The unitary group $U(n)$ acts transitively on $F(n)$. Using this action we obtain an algebraic description for $F(n)$:

$$F(n) = \frac{U(n)}{T} = \frac{U(n)}{U(1) \times \cdots \times U(1)}^n,$$

where $T = U(1) \times \cdots \times U(1)$ is a maximal torus in $U(n)$.

Let $p$ be the tangent space of $F(n)$ in $(T)$. The Lie algebra $u(n)$ is such that

$$u(n) = \{X \in \text{Mat}(n, \mathbb{C}) : X + X^t = 0\} = p \oplus u(1) \oplus \cdots \oplus u(1).$$

**Definition 2.1.** An invariant almost complex structure on $F(n)$ is a linear map $J : p \to p$ such that $J^2 = -I$.

**Example 2.1.** If we consider $F(3) = U(3)$, in this case

$$p = T(F(3)) = \left\{ \begin{pmatrix} 0 & a & b \\ -\bar{a} & 0 & c \\ -\bar{b} & -\bar{c} & 0 \end{pmatrix} : a, b, c, \in \mathbb{C} \right\}.$$

The following linear map is an example of an almost complex structure on $F(3)$

$$\begin{pmatrix} 0 & a & b \\ -\bar{a} & 0 & c \\ -\bar{b} & -\bar{c} & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & (-\sqrt{-1})a & (-\sqrt{-1})b \\ -\sqrt{-1}a & 0 & (-\sqrt{-1})c \\ -\sqrt{-1}b & -\sqrt{-1}c & 0 \end{pmatrix}.$$

There is a natural identification between almost complex structures on $F(n)$ and tournaments with $n$ players.

**Definition 2.2.** A tournament or $n$-tournament $T$, consists of a finite set $T = \{p_1, p_2, \ldots, p_n\}$ of $n$ players, together with a dominance relation, $\to$, that assigns to every pair of players a winner, i.e. $p_i \to p_j$ or $p_j \to p_i$. If $p_i \to p_j$ then we say that $p_i$ beats $p_j$.

A tournament $T$ may be represented by a directed graph in which $T$ is the set of vertices and any two vertices are joined by an oriented edge.

Let $T_1$ be a tournament with $n$ players $\{1, \ldots, n\}$ and $T_2$ another tournament with $m$ players $\{1, \ldots, m\}$. A homomorphism between $T_1$ and $T_2$ is a mapping $\phi : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that

$$s \xrightarrow{T_1} t \implies \phi(s) \xrightarrow{T_2} \phi(t) \text{ or } \phi(s) = \phi(t).$$
When $\phi$ is bijective we said that $T_1$ and $T_2$ are isomorphic.

An $n$-tournament determines a score vector

\[ (s_1, \ldots, s_n), \text{ such that } \sum_{i=1}^{n} s_i = \binom{n}{2}, \]

with components equal the number of games won by each player. Isomorphic tournaments have identical score vectors. Figure 1 shows the isomorphism classes of $n$-tournaments for $n = 2, 3, 4$, together with their score vectors. For $n \geq 5$, there exist non-isomorphic $n$-tournaments with identical score vectors, see Figure 2. The canonical $n$-tournament $T_n$ is defined by setting $i \to j$ if

\[ \begin{align*}
(1) & \quad (0,1) \\
(2) & \quad (0,1,2) \\
(3) & \quad (1,1,1) \\
(4) & \quad (0,1,2,3) \\
(5) & \quad (1,1,1,3) \\
(6) & \quad (0,2,2,2) \\
(7) & \quad (1,1,2,2)
\end{align*} \]

**Figure 1.** Isomorphism classes of $n$-tournaments to $n = 2, 3, 4$.

and only if $i < j$. Up to isomorphism, $T_n$ is the unique $n$-tournament satisfying the following equivalent conditions:

- the dominance relation is transitive, i.e. if $i \to j$ and $j \to k$ then $i \to k$,
- there are no 3-cycles, i.e. closed paths $i_1 \to i_2 \to i_3 \to i_1$, see [M],
- the score vector is $(0,1,2,\ldots,n-1)$.

For each invariant almost complex structure $J$ on $F(n)$, we can associate a $n$-tournament $T(J)$ in the following way: If $J(a_{ij}) = (a'_{ij})$ then $T(J)$ is such that for $i < j$

\[ (i \to j \Leftrightarrow a'_{ij} = \sqrt{-1} a_{ij}) \quad \text{or} \quad (i \leftarrow j \Leftrightarrow a'_{ij} = -\sqrt{-1} a_{ij}), \]

see [MN3].

**Example 2.2.** The tournament in the Figure 3 corresponds to the almost complex structure in the example 2.1
An almost complex structure \( J \) on \( F(n) \) is said to be integrable if \( (F(n), J) \) is a complex manifold. An equivalent condition is the famous Newlander-Nirenberg equation (see [NN]):

\[
\]
for all tangent vectors $X, Y$.

Burstall and Salamon [BS] proved the following result:

**Theorem 2.1.** An almost complex structure $J$ on $F(n)$ is integrable if and only if $T(J)$ is isomorphic to the canonical tournament $T_n$.

Thus, if $T(J)$ contains a 3-cycle then $J$ is not integrable. The almost complex structure of example 2.1 is integrable.

An invariant almost complex structure $J$ on $F(n)$ is called parabolic if there is a permutation $\tau$ of $n$ elements such that the associate tournament $T(J)$ is given, for $i < j$, by

\[
\left( \tau(j) \to \tau(i), \quad \text{if } j - i \text{ is even} \right) \quad \text{or} \quad \left( \tau(i) \to \tau(j), \quad \text{if } j - i \text{ is odd} \right)
\]

Classes (3) and (7) in Figure 1 and (12) in Figure 2 represent the parabolic structures on $F(3), F(4)$ and $F(5)$ respectively.

A $n$-tournament $T$, for $n \geq 3$, is called irreducible or Hamiltonian if it contains a $n$-cycle, i.e. a path

\[
\pi(n) \to \pi(1) \to \pi(2) \to \cdots \to \pi(n-1) \to \pi(n),
\]

where $\pi$ is a permutation of $n$ elements.

A $n$-tournament $T$ is transitive if given three nodes $i, j, k$ of $T$ then

\[
i \to j \quad \text{and} \quad j \to k \implies i \to k.
\]

The canonical tournament is the only one transitive tournament up to isomorphisms.

We consider $\mathbb{C}^n$ equipped with the standard Hermitian inner product, i.e. for $V = (v_1, \ldots, v_n)$ and $W = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we have

\[
\langle V, W \rangle = \sum_{i=1}^{n} v_i \overline{w_i}.
\]

We use the convention

\[
\overline{v_i} = v_i \quad \text{and} \quad \overline{f_{ij}} = f_{ji}.
\]

A frame consists of an ordered set of $n$ vectors $(Z_1, \ldots, Z_n)$, such that $Z_1 \wedge \ldots \wedge Z_n \neq 0$, and it is called unitary, if $\langle Z_i, Z_j \rangle = \delta_{ij}$. The set of unitary frames can be identified with the unitary group.

If we write

\[
dZ_i = \sum_j \omega_{ij} Z_j,
\]

the coefficients $\omega_{ij}$ are the Maurer-Cartan forms of the unitary group $U(n)$. They are skew-Hermitian, i.e.

\[
\omega_{ij} + \omega_{ji} = 0,
\]
and satisfy the equation
\begin{equation}
    d\omega_{ij} = \sum_{k} \omega_{ik} \wedge \omega_{kj}.
\end{equation}

For more details see [ChW].

We may define all left invariant metrics on \((F(n), J)\) by (see [Bl] or [N1])
\begin{equation}
    ds^2_\Lambda = \sum_{i,j} \lambda_{ij} \omega_{ij} \otimes \omega_{ij},
\end{equation}
where \(\Lambda = (\lambda_{ij})\) is a real matrix such that:
\begin{equation}
    \lambda_{ij} > 0, \quad \text{if} \quad i \neq j \quad \lambda_{ij} = 0, \quad \text{if} \quad i = j,
\end{equation}
and the Maurer-Cartan forms \(\omega_{ij}\) are such that
\begin{equation}
    \omega_{ij} \in C^{1,0} \text{ ((1,0) type forms) } \iff \quad \mathcal{T}(j) \rightarrow i.
\end{equation}

Note that, if \(\lambda_{ij} = 1\) for all \(i, j\) in (2.13), then we obtain the normal metric (see [ChE]) induced by the Cartan-Killing form of \(U(n)\).

The metrics (2.13) are called Borel type and they are almost Hermitian for every invariant almost complex structure \(J\), i.e. \(ds^2_\Lambda(JX, JY) = ds^2_\Lambda(X, Y)\), for all tangent vectors \(X, Y\). When \(J\) is integrable \(ds^2_\Lambda\) is said to be Hermitian.

**Definition 2.3.** Let \(J\) be an invariant almost complex structure on \(F(n)\), \(\mathcal{T}(J)\) the associated tournament, and \(ds^2_\Lambda\) an invariant metric. The Kähler form with respect to \(J\) and \(ds^2_\Lambda\) is defined by
\begin{equation}
    \Omega(X, Y) = ds^2_\Lambda(X, JY),
\end{equation}
for any tangent vectors \(X, Y\).

For each permutation \(\tau\) of \(n\) elements, the Kähler form can be write in the following way (see [MN2])
\begin{equation}
    \Omega = -2\sqrt{-1} \sum_{i<j} \mu_{\tau(i)\tau(j)} \omega_{\tau(i)\tau(j)} \wedge \omega_{\tau(i)\tau(j)},
\end{equation}
where
\begin{equation}
    \mu_{\tau(i)\tau(j)} = \varepsilon_{\tau(i)\tau(j)} \lambda_{\tau(i)\tau(j)},
\end{equation}
and
\begin{equation}
    \varepsilon_{ij} = \begin{cases} 
    1 & \text{if } i \rightarrow j \\
    -1 & \text{if } j \rightarrow i \\
    0 & \text{if } i = j
    \end{cases}
\end{equation}

**Definition 2.4.** Let \(J\) be an invariant almost complex structure on \(F(n)\). Then \(F(n)\) is said to be almost Kähler if and only if \(\Omega\) is closed, i.e. \(d\Omega = 0\). If \(J\) is integrable and \(\Omega\) is closed then \(F(n)\) is said to be a Kähler manifold.

The following result was proved by Mo and Negreiros in [MN2].
Theorem 2.2.

\begin{equation}
(2.20) \quad d\Omega = 4 \sum_{i<j<k} C_{\tau(i)\tau(j)\tau(k)} \Psi_{\tau(i)\tau(j)\tau(k)},
\end{equation}

where

\begin{equation}
(2.21) \quad C_{ijk} = \mu_{ij} - \mu_{ik} + \mu_{jk},
\end{equation}

and

\begin{equation}
(2.22) \quad \Psi_{ijk} = \text{Im}(\omega_{ij} \wedge \omega_{ik} \wedge \omega_{jk}).
\end{equation}

We denote by $\mathbb{C}^{p,q}$ the space of complex forms with degree $(p, q)$ on $\mathbb{F}(n)$. Then, for any $i, j, k$, we have either

\begin{equation}
(2.23) \quad \Psi_{ijk} \in \mathbb{C}^{0,3} \oplus \mathbb{C}^{3,0} \quad \text{or} \quad \Psi_{ijk} \in \mathbb{C}^{1,2} \oplus \mathbb{C}^{2,1}
\end{equation}

Definition 2.5. An invariant almost Hermitian metric $ds^2_A$ is said to be $(1,2)$-symplectic if and only if $(d\Omega)^{1,2} = 0$. If $d^*\Omega = 0$ then the metric is said to be cosymplectic.

Figure 4 is included in the known Salamon’s paper [Sa] and it contains a classification of the almost Hermitian structures. This figure provides the following implications

$$\text{Kähler} \implies (1,2)\text{-symplectic} \implies \text{cosymplectic}.$$

For a complete classification see [GH].

The following result due to Mo and Negreiros [MN2], is very useful to study $(1,2)$-symplectic metrics on $\mathbb{F}(n)$:

Theorem 2.3. If $J$ is a $U(n)$-invariant almost complex structure on $\mathbb{F}(n)$, $n \geq 4$, such that $T(J)$ contains one of 4-tournaments in the Figure 5 then $J$ does not admit any invariant $(1,2)$-symplectic metric.

A smooth map $\phi\colon (M, g) \to (N, h)$ between two Riemannian manifolds is said to be harmonic if and only if it is a critical point of the energy functional

\begin{equation}
(2.24) \quad E(\phi) = \frac{1}{2} \int_M |d\phi|^2 v_g,
\end{equation}

where $|d\phi|$ is the Hilbert–Schmidt norm of the linear map $d\phi$, i.e. $\phi$ is harmonic if and only if it satisfies the Euler–Lagrange equations

\begin{equation}
(2.25) \quad \delta E(\phi) = \frac{d}{dt} \bigg|_{t=0} E(\phi_t) = 0
\end{equation}

for all variation $(\phi_t)$ of $\phi$ and $t \in (-\varepsilon, \varepsilon)$ (see [EL]).
3. (1, 2)-Symplectic Structures on $F(3)$ and $F(4)$

It is known that, on $F(3)$ there is a 2-parametric family of Kähler metrics and a 3-parametric family of (1,2)-symplectic metrics corresponding to the non-integrable almost complex structures class. Then each invariant almost complex structure on $F(3)$ admits a (1,2)-symplectic metric, see [ESa], [Bo].
On $F(4)$ there are four isomorphism classes of 4-tournaments or equivalently almost complex structures and the Theorem 2.3 shows that two of them do not admit (1,2)-symplectic metric. The another two classes corresponding to the Kähler and parabolic cases. $F(4)$ has a 3-parametric family of Kähler metrics and a 4-parametric family of (1,2)-symplectic metrics which is not Kähler, see [MN2].

4. (1, 2)-Symplectic Structures on $F(5)$

Figure 2 shows the twelve isomorphism classes of 5-tournaments. The class (1) corresponds to the integrable complex structures and it contains the Kähler metrics. The other classes correspond to non-integrable almost complex structures, in particular the class (11) corresponds to the parabolic structure.

To the remain classes we have the following result:

**Theorem 4.1.** Between the classes of 5-tournaments (Figure 2), the only ones that admit (1,2)-symplectic metrics, different to the Kähler and parabolic, are (7) and (9).

**Proof.** We use the Theorem 2.3 to prove that (2), (3), (4), (5), (6), (8), (10) and (11) do not admit (1,2)-symplectic metric. It is easy to see that: (2) contains $T_1$ formed by the vertices $1,2,3,4$; (3) contains $T_1$ formed by the vertices $2,3,4,5$; (4) contains $T_2$ formed by the vertices $1,2,3,4$; (5) contains $T_2$ formed by the vertices $2,3,4,5$; (6) contains $T_2$ formed by the vertices $1,3,4,5$; (8) contains $T_2$ formed by the vertices $2,3,4,5$; (10) contains $T_1$ formed by the vertices $1,2,3,4$ and (11) contains $T_2$ formed by the vertices $1,2,3,4$. Then neither of them admit (1,2)-symplectic metric.

Using formulas (2.20)-(2.23), we obtain that (7) admits (1,2)-symplectic metric if and only if $\Lambda = (\lambda_{ij})$ satisfies the linear system

\[
\begin{align*}
\lambda_{12} - \lambda_{13} + \lambda_{23} &= 0 \\
\lambda_{12} - \lambda_{14} + \lambda_{24} &= 0 \\
\lambda_{13} - \lambda_{14} + \lambda_{34} &= 0 \\
\lambda_{23} - \lambda_{24} + \lambda_{34} &= 0 \\
\lambda_{23} - \lambda_{25} + \lambda_{35} &= 0 \\
\lambda_{24} - \lambda_{25} + \lambda_{45} &= 0 \\
\lambda_{34} - \lambda_{35} + \lambda_{45} &= 0
\end{align*}
\]

Then (7) admits (1,2)-symplectic metric if and only if $\Lambda = (\lambda_{ij})$ satisfies

\[
\begin{align*}
\lambda_{13} &= \lambda_{12} + \lambda_{23} \\
\lambda_{14} &= \lambda_{12} + \lambda_{23} + \lambda_{34} \\
\lambda_{24} &= \lambda_{23} + \lambda_{34} \\
\lambda_{25} &= \lambda_{23} + \lambda_{34} + \lambda_{45} \\
\lambda_{35} &= \lambda_{34} + \lambda_{45}
\end{align*}
\]
Similarly, we obtain that (9) admit (1,2)-symplectic metric if and only if \( \Lambda = (\lambda_{ij}) \) satisfies
\[
\begin{align*}
\lambda_{13} &= \lambda_{12} + \lambda_{23} \\
\lambda_{14} &= \lambda_{12} + \lambda_{23} + \lambda_{34} \\
\lambda_{24} &= \lambda_{23} + \lambda_{34} \\
\lambda_{25} &= \lambda_{12} + \lambda_{15} \\
\lambda_{35} &= \lambda_{34} + \lambda_{45}
\end{align*}
\]

Now we can write the respective matrices
\[
\Lambda(7) = \begin{pmatrix}
0 & \lambda_{12} & \lambda_{12} + \lambda_{23} & \lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{15} \\
\lambda_{12} & 0 & \lambda_{23} & \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} + \lambda_{45} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34} + \lambda_{45} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} & \lambda_{34} & 0 & \lambda_{45} \\
\lambda_{15} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} & \lambda_{45} & 0
\end{pmatrix}
\]
\[
\Lambda(9) = \begin{pmatrix}
0 & \lambda_{12} & \lambda_{12} + \lambda_{23} & \lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{15} \\
\lambda_{12} & 0 & \lambda_{23} & \lambda_{23} + \lambda_{34} & \lambda_{12} + \lambda_{15} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34} + \lambda_{45} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} & \lambda_{34} & 0 & \lambda_{45} \\
\lambda_{15} & \lambda_{12} + \lambda_{15} & \lambda_{34} + \lambda_{45} & \lambda_{45} & 0
\end{pmatrix}
\]

The Theorem 4.1 says that \( F(n) \) admits (1,2)-symplectic metrics, different to the Kähler and parabolic, if and only if \( n \geq 5 \).

5. (1, 2)-Symplectic Structures on \( F(6) \)

There are 56 isomorphism classes of 6-tournaments (see [M]), which are presented in Figures 6, 7 and 8. Again, the class (1) corresponds to the integrable complex structures. The other classes correspond to non-integrable almost complex structures, and the class (52) corresponds to the parabolic structure.

In this case we have the following result

**Theorem 5.1.** Between the classes of 6-tournaments (Figure 6, 7 and 8), the only ones that admit (1,2)-symplectic metrics, different to the Kähler and parabolic, are (19), (31), (37) and (55).

**Proof.** We use the Theorem 2.3 to prove that each of the classes of 6-tournaments different to the (1), (19), (31), (37), (52) and (55) does not admit (1,2)-symplectic metrics:

- (2) contains \( T_1 \) formed by the vertices 1,2,3,4.
- (3) contains \( T_2 \) formed by the vertices 1,2,3,4.
- (4) contains \( T_1 \) formed by the vertices 1,2,3,5.
- (5) contains \( T_2 \) formed by the vertices 2,3,4,5.
Figure 6. Isomorphism classes of 6-tournaments

- (6) contains $T_2$ formed by the vertices 1, 2, 3, 4.
- (7) contains $T_1$ formed by the vertices 1, 2, 3, 4.
- (8) contains $T_1$ formed by the vertices 1, 2, 3, 4.
Figure 7. Isomorphism classes of 6-tournaments

- (9) contains $T_1$ formed by the vertices 1,2,3,4.
- (10) contains $T_1$ formed by the vertices 1,2,3,4.
- (11) contains $T_2$ formed by the vertices 1,2,3,4.
Figure 8. Isomorphism classes of 6-tournaments

- (12) contains $T_1$ formed by the vertices 2,3,5,6.
- (13) contains $T_2$ formed by the vertices 3,4,5,6.
- (14) contains $T_2$ formed by the vertices 3,4,5,6.
- (15) contains $T_2$ formed by the vertices 2,3,4,5.
- (16) contains $T_2$ formed by the vertices 1,2,3,4.
- (17) contains $T_2$ formed by the vertices 3,4,5,6.
- (18) contains $T_2$ formed by the vertices 3,4,5,6.
- (20) contains $T_2$ formed by the vertices 2,3,4,5.
- (21) contains $T_2$ formed by the vertices 2,3,4,5.
- (22) contains $T_1$ formed by the vertices 1,2,3,5.
- (23) contains $T_1$ formed by the vertices 1,2,3,5.
• (24) contains \( T_2 \) formed by the vertices 1,2,3,4.
• (25) contains \( T_2 \) formed by the vertices 1,2,3,4.
• (26) contains \( T_2 \) formed by the vertices 3,4,5,6.
• (27) contains \( T_2 \) formed by the vertices 2,3,4,5.
• (28) contains \( T_2 \) formed by the vertices 3,4,5,6.
• (29) contains \( T_2 \) formed by the vertices 2,3,4,5.
• (30) contains \( T_2 \) formed by the vertices 2,3,4,5.
• (31) contains \( T_2 \) formed by the vertices 2,3,4,5.
• (32) contains \( T_1 \) formed by the vertices 1,2,3,4.
• (33) contains \( T_2 \) formed by the vertices 3,4,5,6.
• (34) contains \( T_2 \) formed by the vertices 3,4,5,6.
• (35) contains \( T_2 \) formed by the vertices 2,3,4,5.
• (36) contains \( T_2 \) formed by the vertices 1,2,3,4.
• (37) contains \( T_2 \) formed by the vertices 1,2,3,4.
• (38) contains \( T_2 \) formed by the vertices 1,2,3,4.
• (39) contains \( T_2 \) formed by the vertices 3,4,5,6.
• (40) contains \( T_2 \) formed by the vertices 3,4,5,6.
• (41) contains \( T_2 \) formed by the vertices 3,4,5,6.
• (42) contains \( T_2 \) formed by the vertices 3,4,5,6.
• (43) contains \( T_2 \) formed by the vertices 3,4,5,6.
• (44) contains \( T_2 \) formed by the vertices 3,4,5,6.
• (45) contains \( T_2 \) formed by the vertices 1,2,3,4.
• (46) contains \( T_2 \) formed by the vertices 2,3,5,6.
• (47) contains \( T_2 \) formed by the vertices 1,2,3,4.
• (48) contains \( T_2 \) formed by the vertices 2,3,4,5.
• (49) contains \( T_2 \) formed by the vertices 1,2,3,4.
• (50) contains \( T_2 \) formed by the vertices 1,2,3,4.
• (51) contains \( T_2 \) formed by the vertices 1,2,3,4.
• (52) contains \( T_2 \) formed by the vertices 1,2,3,4.
• (53) contains \( T_2 \) formed by the vertices 1,2,3,4.
• (54) contains \( T_2 \) formed by the vertices 1,2,3,4.
• (55) contains \( T_2 \) formed by the vertices 1,2,3,4.
• (56) contains \( T_2 \) formed by the vertices 1,2,3,4.

By making similar computations to we made in the proof of Theorem 4.1 we obtain:

• The class (19) admits (1,2)-symplectic metric if and only if the elements of corresponding matrix \( \Lambda_{(19)} = (\lambda_{ij}) \) satisfy the following system of linear equations

\[
\begin{align*}
\lambda_{12} - \lambda_{13} + \lambda_{23} &= 0 \\
\lambda_{12} - \lambda_{15} + \lambda_{25} &= 0 \\
\lambda_{13} - \lambda_{15} + \lambda_{35} &= 0 \\
\lambda_{23} - \lambda_{24} + \lambda_{34} &= 0 \\
\lambda_{23} - \lambda_{26} + \lambda_{36} &= 0 \\
\lambda_{24} - \lambda_{26} + \lambda_{46} &= 0 \\
\lambda_{34} - \lambda_{35} + \lambda_{45} &= 0 \\
\lambda_{35} - \lambda_{36} + \lambda_{56} &= 0 \\
\lambda_{45} - \lambda_{46} + \lambda_{56} &= 0.
\end{align*}
\]
Then the metric $ds^2_{\Lambda_{(19)}}$ is (1,2)-symplectic if and only if

\[
\begin{align*}
\lambda_{13} &= \lambda_{12} + \lambda_{23} \\
\lambda_{14} &= \lambda_{12} + \lambda_{23} + \lambda_{34} \\
\lambda_{15} &= \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} \\
\lambda_{24} &= \lambda_{23} + \lambda_{34} \\
\lambda_{25} &= \lambda_{23} + \lambda_{34} + \lambda_{45}.
\end{align*}
\]

In similar way the class (31) admits (1,2)-symplectic metric if and only if the elements of the corresponding matrix $\Lambda_{(31)} = (\lambda_{ij})$ satisfy the following relations

\[
\begin{align*}
\lambda_{13} &= \lambda_{12} + \lambda_{23} \\
\lambda_{14} &= \lambda_{12} + \lambda_{23} + \lambda_{34} \\
\lambda_{15} &= \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} \\
\lambda_{24} &= \lambda_{23} + \lambda_{34} \\
\lambda_{25} &= \lambda_{23} + \lambda_{34} + \lambda_{45}.
\end{align*}
\]

Similarly, the class (37) admits (1,2)-symplectic metric if and only if the elements of the corresponding matrix $\Lambda_{(37)} = (\lambda_{ij})$ satisfy the following relations

\[
\begin{align*}
\lambda_{14} &= \lambda_{12} + \lambda_{25} + \lambda_{45} \\
\lambda_{15} &= \lambda_{12} + \lambda_{25} \\
\lambda_{16} &= \lambda_{12} + \lambda_{25} + \lambda_{45} + \lambda_{46} \\
\lambda_{23} &= \lambda_{12} + \lambda_{13} \\
\lambda_{24} &= \lambda_{25} + \lambda_{45}.
\end{align*}
\]

Finally, the class (55) admits (1,2)-symplectic metric if and only if the elements of the corresponding matrix $\Lambda_{(55)} = (\lambda_{ij})$ satisfy the following relations

\[
\begin{align*}
\lambda_{13} &= \lambda_{12} + \lambda_{25} + \lambda_{35} \\
\lambda_{15} &= \lambda_{12} + \lambda_{25} \\
\lambda_{16} &= \lambda_{14} + \lambda_{46} \\
\lambda_{23} &= \lambda_{25} + \lambda_{35} \\
\lambda_{24} &= \lambda_{12} + \lambda_{14}.
\end{align*}
\]

The matrices $\Lambda_{(19)}, \Lambda_{(31)}, \Lambda_{(37)}$ and $\Lambda_{(55)}$ corresponding to the classes (19), (31), (37) and (55) are presented on the end of this paper.

6. (1, 2)-Symplectic Structures on $F(7)$

This case has a problem because it is not known any collection of tournament drawings for $n \geq 7$. The collection of tournaments drawings of $n = 2, 3, 4, 5, 6$, is contained in the Moon’s book [M].
There are 456 isomorphism classes of 7-tournaments. In the Dias’s M. Sc. Thesis [D] was obtained a representant matrix of each class of 7-tournament. The matrix $M(T) = (a_{ij})$ of the tournament $T$ is defined by

$$a_{ij} = \begin{cases} 0, & \text{if } j \xrightarrow{T} i \\ 1, & \text{if } i \xrightarrow{T} j. \end{cases}$$

Obviously, it has the matrix is equivalent to have the tournament drawing.

We used the matrices generated in [D] together with the Digraph computer program, created by Professor Davide Carlo Demaria, in order to know which 7-tournaments contain the tournaments in Figure 5. Table 1 shows the matrices of the 7-tournaments which admit (1,2)-symplectic metric. Using the matrices

$$
\begin{array}{cccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0
\end{array}
$$

$$
\begin{array}{cccccccc}
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

$$
\begin{array}{cccccccc}
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

$$
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Table 1. Matrices of the 7-tournaments which admit (1,2)-symplectic metric

in the Table 1 we construct the 7-tournament drawings which admit (1,2)-symplectic metric. Figures 9 and 10 show this 7-tournaments. Class (1) in the Figure 9 represents the integrable structures and the class (10) in Figure 10 corresponds to the parabolic structures. To the remain classes we have the following result.
Figure 9. Isomorphism classes of 7-tournaments which admit (1,2)-symplectic metric

Theorem 6.1. The classes of 7-tournaments (2) through (9) in the Figures 9 and 10 admit (1, 2)-symplectic metrics, different to the Kähler and parabolic.
Figure 10. Isomorphism classes of 7-tournaments which admit (1,2)-symplectic metric

Proof. The proof is made through a long calculation similar to the proof of Theorem 4.1.

The matrices $\Lambda^{(2)}$ through $\Lambda^{(9)}$ corresponding to the classes (2) through (9) are presented on the end of this paper.

Wolf and Gray [WG] proved that the normal metric on $F(n)$ is not (1,2)-symplectic for $n \geq 4$. Our results give a simple proof of this fact to $n = 5, 6, 7$.

7. Harmonic Maps

In this section we construct new examples of harmonic maps using the following result due to Lichnerowicz [L]:

\[\text{(8)} \cdot 2 \quad 3 \quad 1 \quad 4 \quad 5 \quad 6 \quad 7 \quad (2,2,3,3,3,4,4) \]

\[\text{(9)} \cdot 2 \quad 3 \quad 1 \quad 4 \quad 5 \quad 6 \quad 7 \quad (2,3,3,3,3,3,4) \]

\[\text{(10)} \cdot 2 \quad 3 \quad 1 \quad 4 \quad 5 \quad 6 \quad 7 \quad (3,3,3,3,3,3,3) \]
Theorem 7.1. Let \( \phi : (M, g, J_1) \rightarrow (N, h, J_2) \) be a \( \pm \) holomorphic map between almost Hermitian manifolds where \( M \) is cosymplectic and \( N \) is \( (1,2) \)-symplectic. Then \( \phi \) is harmonic. (\( \phi \) is \( \pm \) holomorphic if and only if \( d\phi \circ J_1 = \pm J_2 \circ d\phi \)).

In order to construct harmonic maps \( \phi : M^2 \rightarrow F(n) \) using the theorem above, we need to know examples of holomorphic maps. Then we use the following construction due to Eells and Wood [EW].

Let \( h : M^2 \rightarrow \mathbb{C}P^{n-1} \) be a full holomorphic map (\( h \) is full if \( h(M) \) is not contained in none \( \mathbb{C}P^k \), for all \( k < n-1 \)). We can lift \( h \) to \( \mathbb{C}^n \), i.e. for every \( p \in M \) we can find a neighborhood of \( p \), \( U \subset M \), such that \( h(U) = (u_0, \ldots, u_{n-1}) : M^2 \supset U \rightarrow \mathbb{C}^n \) satisfies \( h(z) = [h_U(z)] = [(u_0(z), \ldots, u_{n-1}(z))] \).

We define the \( k \)-th associate curve of \( h \) by

\[
O_k : M^2 \rightarrow \mathbb{G}_{k+1}(\mathbb{C}^n)
\]

for \( 0 \leq k \leq n-1 \). And we consider

\[
h_k : M^2 \rightarrow \mathbb{C}P^{n-1}
\]

for \( 0 \geq k \geq n-1 \).

The following theorem, due to Eells and Wood ([EW]), is very important because it gives the classification of the harmonic maps from \( S^2 \sim \mathbb{C}P^1 \) into a projective space \( \mathbb{C}P^{n-1} \).

Theorem 7.2. For each \( k \in \mathbb{N}, 0 \leq k \leq n-1 \), \( h_k \) is harmonic. Furthermore, given \( \phi : (\mathbb{C}P^1, g) \rightarrow (\mathbb{C}P^{n-1}, \text{Killing metric}) \) a full harmonic map, then there are unique \( k \) and \( h \) such that \( \phi = h_k \).

This theorem provides in a natural way the following holomorphic maps

\[
\Psi : M^2 \rightarrow F(n)
\]

\[
z \mapsto (h_0(z), \ldots, h_{n-1}(z)),
\]

called by Eells–Wood’s map (see [N2]).

We called \( \mathcal{M}_n \) the set of \((1,2)\)-symplectic metrics on \( F(n) \), for \( n = 5, 6 \) and \( 7 \) characterized in the sections above. Using Theorem 7.1 we obtain the following result

Theorem 7.3. Let \( \phi : M^2 \rightarrow (F(n), g), g \in \mathcal{M}_n \) a holomorphic map. Then \( \phi \) is harmonic.

In addition for maps from a flag manifold into a flag manifold we obtain the following result

Proposition 7.1. Let \( \phi : (F(l), g) \rightarrow (F(k), h) \) a holomorphic map, with \( g \in \mathcal{M}_l \) and \( h \in \mathcal{M}_k \). Then \( \phi \) is harmonic.
\[
\lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56}
\]

\[
\begin{array}{ccccccc}
\lambda_{12} & 0 & \lambda_{23} & \lambda_{34} & \lambda_{45} & \lambda_{56} & 0 \\
0 & \lambda_{12} & \lambda_{23} + \lambda_{24} & \lambda_{34} + \lambda_{45} & \lambda_{56} & 0 & \\
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{24} + \lambda_{34} & \lambda_{34} + \lambda_{45} & \lambda_{56} & 0 & \\
\lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{23} + \lambda_{24} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{34} + \lambda_{45} + \lambda_{56} & 0 & \\
\lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{23} + \lambda_{24} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{34} + \lambda_{45} + \lambda_{56} & 0 & \\
\lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{23} + \lambda_{24} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{34} + \lambda_{45} + \lambda_{56} & 0 & \\
\end{array}
\]
\[
\begin{align*}
\Lambda_{(ii)} &= \\
& \begin{cases}
\lambda_{12} & \text{if } i = j = 1 \\
\lambda_{12} + \lambda_{23} & \text{if } i = j = 2 \\
\lambda_{14} & \text{if } i = j = 3 \\
\lambda_{23} & \text{if } i = j = 4 \\
\lambda_{34} & \text{if } i = j = 5 \\
\lambda_{45} & \text{if } i = j = 6 \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]
\begin{align*}
\Lambda_{(37)} & = \begin{pmatrix}
\lambda_{12} & \lambda_{13} & 0 \\
\lambda_{12} & \lambda_{12} + \lambda_{13} & 0 \\
\lambda_{12} & 0 & \lambda_{25} + \lambda_{45} + \lambda_{6} \\
\lambda_{12} + \lambda_{25} & \lambda_{25} + \lambda_{45} + \lambda_{6} & 0 \\
\lambda_{12} + \lambda_{25} & 0 & \lambda_{13} + \lambda_{14} + \lambda_{15} + \lambda_{16} + \lambda_{25} + \lambda_{45} + \lambda_{6}
\end{pmatrix}
\end{align*}
\[ \Lambda_{(55)} = \begin{pmatrix}
0 & \lambda_{12} & \lambda_{12} + \lambda_{25} + \lambda_{35} & \lambda_{14} & \lambda_{12} + \lambda_{25} & \lambda_{14} + \lambda_{46} \\
\lambda_{12} & 0 & \lambda_{25} + \lambda_{35} & \lambda_{12} + \lambda_{14} & \lambda_{25} & \lambda_{12} + \lambda_{14} + \lambda_{46} \\
\lambda_{12} + \lambda_{25} + \lambda_{35} & \lambda_{25} + \lambda_{35} & 0 & \lambda_{36} + \lambda_{46} & \lambda_{35} & \lambda_{36} \\
\lambda_{14} & \lambda_{12} + \lambda_{14} & \lambda_{36} + \lambda_{46} & 0 & \lambda_{35} + \lambda_{36} + \lambda_{46} & \lambda_{35} \\
\lambda_{12} + \lambda_{25} & \lambda_{25} & \lambda_{35} & \lambda_{35} + \lambda_{36} + \lambda_{46} & 0 & \lambda_{35} + \lambda_{36} \\
\lambda_{14} + \lambda_{46} & \lambda_{12} + \lambda_{14} + \lambda_{46} & \lambda_{36} & \lambda_{36} + \lambda_{46} & \lambda_{35} + \lambda_{36} & 0
\end{pmatrix} \]
\( \Lambda_{(2)} = \left( \begin{array}{ccccccccccc}
0 & \lambda_{12} & \lambda_{12} + \lambda_{23} & \lambda_{23} & \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} \\
\lambda_{12} & 0 & \lambda_{23} & \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} + \lambda_{67} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} + \lambda_{67} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45} + \lambda_{56} & \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{45} + \lambda_{56} + \lambda_{67} + \lambda_{67} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} & \lambda_{45} & 0 & \lambda_{56} & \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} + \lambda_{67} \\
\lambda_{17} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} & \lambda_{67} & 0
\end{array} \right) \)
\[ \Lambda_{(3)} = \begin{pmatrix}
0 & \lambda_{12} & \lambda_{12} + \lambda_{23} & \lambda_{12} + \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{12} + \lambda_{12} + \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{12} + \lambda_{12} + \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{17} \\
\lambda_{12} & 0 & \lambda_{23} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{23} + \lambda_{23} + \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{23} + \lambda_{23} + \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{17} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & 0 & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{34} + \lambda_{34} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{34} + \lambda_{34} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{17} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{45} + \lambda_{45} + \lambda_{45} + \lambda_{56} + \lambda_{56} + \lambda_{67} & \lambda_{45} + \lambda_{45} + \lambda_{45} + \lambda_{56} + \lambda_{56} + \lambda_{67} & \lambda_{17} \\
\lambda_{17} & \lambda_{12} + \lambda_{17} + \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} & \lambda_{67} & \lambda_{67} \\
\end{pmatrix} \]
\[
A_{(4)} = \begin{pmatrix}
0 & \lambda_{12} & \lambda_{12} + \lambda_{23} & \lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{17} \\
\lambda_{12} & 0 & \lambda_{23} & \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{12} + \lambda_{17} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{12} + \lambda_{17} + \lambda_{23} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45} + \lambda_{56} & \lambda_{12} + \lambda_{17} + \lambda_{23} + \lambda_{56} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} & \lambda_{45} & 0 & \lambda_{56} & \lambda_{56} + \lambda_{57} \\
\lambda_{17} & \lambda_{12} + \lambda_{17} & \lambda_{12} + \lambda_{17} + \lambda_{23} & \lambda_{12} + \lambda_{17} + \lambda_{23} + \lambda_{45} & \lambda_{12} + \lambda_{17} + \lambda_{23} + \lambda_{45} + \lambda_{56} & \lambda_{12} + \lambda_{17} + \lambda_{23} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{57} \\
\end{pmatrix}
\]
\( \Lambda_{(5)} = \)

\[
\begin{pmatrix}
0 & \lambda_{12} & \lambda_{12} + \lambda_{23} & \lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{17} \\
\lambda_{12} & 0 & \lambda_{23} & \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{12} + \lambda_{17} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{12} + \lambda_{17} + \lambda_{23} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45} + \lambda_{56} & \lambda_{12} + \lambda_{17} + \lambda_{23} + \lambda_{34} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} & \lambda_{45} & 0 & \lambda_{56} & \lambda_{56} + \lambda_{67} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{45} + \lambda_{56} & \lambda_{56} & 0 & \lambda_{67} \\
\lambda_{17} & \lambda_{12} + \lambda_{17} & \lambda_{12} + \lambda_{17} + \lambda_{23} & \lambda_{12} + \lambda_{17} + \lambda_{23} + \lambda_{34} & \lambda_{56} + \lambda_{67} & \lambda_{67} & 0
\end{pmatrix}
\]
\[ \Lambda_{(6)} = \begin{pmatrix}
0 & \lambda_{12} & \lambda_{12} + \lambda_{23} & \lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{17} + \lambda_{67} & \lambda_{17} \\
\lambda_{12} & 0 & \lambda_{23} & \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{56} & \lambda_{12} + \lambda_{17} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45} + \lambda_{56} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} & \lambda_{45} & 0 & \lambda_{56} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} \\
\lambda_{17} + \lambda_{67} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{45} + \lambda_{56} & \lambda_{56} & 0 & \lambda_{67} \\
\lambda_{17} & \lambda_{12} + \lambda_{17} & \lambda_{12} + \lambda_{17} + \lambda_{56} + \lambda_{67} & \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} & \lambda_{67} & 0
\end{pmatrix} \]
\[ \Lambda_{17} = \begin{pmatrix}
0 & \lambda_{12} & \lambda_{12} + \lambda_{23} & \lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{17} + \lambda_{67} & \lambda_{17} \\
\lambda_{12} & 0 & \lambda_{23} & \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{17} + \lambda_{17} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{12} + \lambda_{17} + \lambda_{23} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} & \lambda_{34} & 0 & \lambda_{45} + \lambda_{56} + \lambda_{67} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} & \lambda_{45} & 0 & \lambda_{56} + \lambda_{67} \\
\lambda_{17} + \lambda_{67} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{56} & 0 \\
\lambda_{17} + \lambda_{17} & \lambda_{12} + \lambda_{17} + \lambda_{23} & \lambda_{12} + \lambda_{17} + \lambda_{23} + \lambda_{67} & \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} & 0 \\
\lambda_{17} & \lambda_{12} + \lambda_{17} & \lambda_{12} + \lambda_{17} + \lambda_{23} + \lambda_{67} & \lambda_{56} + \lambda_{67} & \lambda_{67} & 0
\end{pmatrix} \]
\[
\Lambda_{(8)} = \\
\begin{pmatrix}
0 & \lambda_{12} & \lambda_{12} + \lambda_{23} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{12} + \lambda_{17} + \lambda_{67} & \lambda_{17} \\
\lambda_{12} & 0 & \lambda_{23} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{12} + \lambda_{17} & \lambda_{12} + \lambda_{17} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & 0 & \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} & \lambda_{34} + \lambda_{45} & \lambda_{45} & \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{45} + \lambda_{56} + \lambda_{67} \\
\lambda_{12} + \lambda_{17} + \lambda_{67} & \lambda_{12} + \lambda_{17} + \lambda_{67} & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} \\
\lambda_{17} & \lambda_{12} + \lambda_{17} + \lambda_{67} & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} \\
\end{pmatrix}
\]
### References


$$
\lambda_{1(0)} = \begin{pmatrix}
0 & \lambda_{12} & \lambda_{12} + \lambda_{23} & \lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{17} + \lambda_{67} & \lambda_{17} \\
\lambda_{12} & 0 & \lambda_{23} & \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{12} + \lambda_{17} + \lambda_{67} & \lambda_{12} + \lambda_{17} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{12} + \lambda_{17} + \lambda_{67} & \lambda_{12} + \lambda_{17} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45} + \lambda_{56} & \lambda_{12} + \lambda_{17} + \lambda_{67} & \lambda_{12} + \lambda_{17} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} & \lambda_{45} & 0 & \lambda_{56} & \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} \\
\lambda_{17} + \lambda_{67} & \lambda_{12} + \lambda_{17} + \lambda_{67} & \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{45} + \lambda_{56} & \lambda_{56} & 0 & \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} \\
\lambda_{17} & \lambda_{12} + \lambda_{17} & \lambda_{12} + \lambda_{17} + \lambda_{23} & \lambda_{45} + \lambda_{56} & \lambda_{56} & \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} \\
\lambda_{17} & \lambda_{12} + \lambda_{17} & \lambda_{12} + \lambda_{17} + \lambda_{23} + \lambda_{67} & \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67}
\end{pmatrix}
$$


(Recibido en octubre de 2000)

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