ALMOST NORMALITY AND MILD NORMALITY
OF THE TYCHONOFF PLANK

LUTFI N. KALANTAN

Abstract: The Tychonoff Plank is a popular example of the fact normality is not hereditary. We will show that it is mildly normal but not almost normal.

The Tychonoff plank $X = (\omega_1 + 1 \times \omega + 1) \setminus \{ (\omega_1, \omega) \}$ is a famous example of a $T_{3\frac{1}{2}}$-space which is not normal, see [1]. It is also a famous example of the fact that normality is not hereditary, see [1]. In this paper, we will show that the Tychonoff plank is mildly normal but not almost normal. We will denote an order pairs by $(x, y)$, the set of positive integers by $\mathbb{N}$ and the set of all real numbers by $\mathbb{R}$.

Definition 1. A subset $A$ of a topological space $X$ is called regularly closed (called also, closed domain) if $A = \overline{\text{int} A}$. Two subsets $A$ and $B$ in a topological space $X$ are said to be separated if there exist two disjoint open subsets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$. \[ \square \]

Definition 2. A topological space $X$ is called mildly normal (called also $\kappa$-normal) if any two disjoint regularly closed subsets $A$ and $B$ of $X$, can be separated. \[ \square \]

Received: March 13, 2004; Revised: May 9, 2004.

AMS Subject Classification: 54D15, 54B10.
Keywords: $\kappa$-normal; mildly normal; almost normal; regularly closed; normal.
In [2], Shchepin introduced the notion of \( \kappa \)-normal property. He required regularity in his definition. In [3], Singal and Singal introduced the notion of mildly normal property. They did not require regularity.

Let \( \omega \) be the first infinite ordinal and \( \omega_1 \) be the first uncountable ordinal with their usual order topology. Consider the product space \( \omega + 1 \times \omega + 1 \). The Tychonoff Plank is the subspace \( X = (\omega + 1 \times \omega + 1) \setminus \{\langle \omega_1, \omega \rangle\} \). Write \( X = A \cup B \cup C \), where \( A = \{\omega_1\} \times \omega \), \( B = \omega_1 \times \{\omega\} \), and \( C = X \setminus (A \cup B) \). Let \( p_1 : \omega_1 + 1 \times \omega + 1 \to \omega_1 + 1 \) and \( p_2 : \omega_1 + 1 \times \omega + 1 \to \omega + 1 \) be the natural projections. To show that \( X \) is mildly normal, we need the following lemma:

**Lemma 1.** If \( H \) and \( K \) are closed disjoint unseparated subsets of \( X \), then either \( (p_1(H \cap B) \) is unbounded and \( p_2(K \cap A) \) is unbounded) or \( (p_1(K \cap B) \) is unbounded and \( p_2(H \cap A) \) is unbounded).

**Proof:** Let \( H \) and \( K \) be any closed disjoint unseparated subsets of \( X \). Suppose that the conclusion is false. This gives us that \((p_1(H \cap B) \) is bounded or \( p_2(K \cap A) \) is bounded) and \( (p_1(K \cap B) \) is bounded and \( p_2(H \cap A) \) is bounded). This gives us the following four cases:

1. \( p_1(H \cap B) \) is bounded and \( p_2(H \cap A) \) is bounded.
2. \( p_1(H \cap B) \) is bounded and \( p_1(K \cap B) \) is bounded.
3. \( p_2(K \cap A) \) is bounded and \( p_2(H \cap A) \) is bounded.
4. \( p_2(K \cap A) \) is bounded and \( p_1(K \cap B) \) is bounded.

**Case 1:** \( p_1(H \cap B) \) is bounded and \( p_2(H \cap A) \) is bounded. Let \( \gamma \) be the least upper bound of \( p_1(H \cap B) \) and \( m \) be the least upper bound of \( p_2(H \cap A) \). In the space \( Y = \omega_1 + 1 \times \omega + 1 \supset X \) we have that \( \langle \omega_1, \omega \rangle \notin \overline{Y} \). Because if \( \langle \omega_1, \omega \rangle \in \overline{Y} \), then for each \( \alpha < \omega_1 \) and for each \( n < \omega \), we have \((\alpha, \omega_1] \times (n, \omega]\) \cap H \neq \emptyset \). Pick \( k > m \) and \( \alpha > \gamma \). Pick \( \langle \alpha_1, k_1 \rangle \in ((\alpha, \omega_1] \times (k, \omega]) \cap H \). Pick \( \langle \alpha_2, k_2 \rangle \in ((\alpha_1, \omega_1] \times (k_1, k_2]) \cap H \). Observe that \( \alpha_1 < \alpha_2 \) and \( k_1 < k_2 \). If \( l \geq 3 \), \( l < \omega \), and \( \langle \alpha_1, k_1 \rangle, \ldots, \langle \alpha_l, k_l \rangle \) are all picked such that \( \alpha_1 < \alpha_2 < \ldots < \alpha_l \) and \( k_1 < k_2 < \ldots < k_l \). Then pick \( \langle \alpha_{l+1}, k_{l+1} \rangle \in ((\alpha_l, \omega_1] \times (k_l, \omega]) \cap H \). By induction, we get a countably infinite sequence \( \{\langle \alpha_i, k_i \rangle : i \in \mathbb{N}\} \) such that \( \alpha_i < \alpha_{i+1} \) and \( k_i < k_{i+1} \) for each \( i \in \mathbb{N} \). Since \( \omega_1 \) has uncountable cofinality, then there exists a limit ordinal \( \beta < \omega_1 \) such that \( \langle \beta, \omega \rangle \) is a limit point of the sequence \( \{\langle \alpha_i, k_i \rangle : i \in \mathbb{N}\} \subseteq H \). Hence \( \langle \beta, \omega \rangle \notin \overline{Y} = H \). This means that \( \langle \beta, \omega \rangle \in H \cap B \) with \( \gamma < \beta \) which is a contradiction because \( \gamma \) is the least upper bound. Therefore, \( H \) is closed in \( Y \). Now, let \( K^* = K \cup \{\omega_1, \omega\} \). Then \( K^* \) is closed in \( Y \) which is disjoint from \( H \).
Since $Y$ is normal, being a $T_2$-compact space, then $H$ and $K^*$ can be separated in $Y$ by two disjoint open sets, say $U$ and $V$ with $H \subseteq U$ and $K^* \subseteq V$. Now, the two $X$-open sets $U$ and $V \cap X$ are disjoint with $H \subseteq U$ and $K \subseteq V \cap X$. So, $H$ and $K$ are separated, which is a contradiction.

**Case 4:** $p_2(K \cap A)$ is bounded and $p_1(K \cap B)$ is bounded. This case is similar to Case 1.

**Case 2:** $p_1(H \cap B)$ is bounded and $p_1(K \cap B)$ is bounded. Let $\gamma_1$ be the least upper bound for $p_1(H \cap B)$ and $\gamma_2$ be the least upper bound for $p_1(K \cap B)$. For each $n \in p_2(K \cap A)$, there exists an $\alpha_n < \omega_1$ such that the open set $V_n = (\alpha_n, \omega_1] \times \{n\}$ is disjoint from $H$. For each $m \in p_2(H \cap A)$, there exists a $\beta_m < \omega_1$ such that the open set $U_m = (\beta_m, \omega_1] \times \{m\}$ is disjoint from $K$. Now, the set $\{\gamma_1, \gamma_2, \alpha_n, \beta_m : n \in p_2(K \cap A), m \in p_2(H \cap A)\}$ is a countable subset of $\omega_1$. Pick an upper bound $\xi$ of it. Now, observe that the set $D = \{(\alpha, k) \in H \cup K : \xi \leq \alpha < \omega_1$ and $k \notin p_2(K \cap A) \cup p_2(H \cap A)\}$ is countable. So, pick an upper bound $\zeta$ of the set $\{\alpha : (\alpha, k) \in D$ for some $k < \omega\}$ with $\xi \leq \zeta$. Let $\eta = \zeta + 1$. We have that $(\eta, \omega_1] \times \{n\} \subseteq V_n$ for each $n \in p_2(K \cap A)$ and $(\eta, \omega_1] \times \{m\} \subseteq U_m$ for each $m \in p_2(H \cap A)$. Thus $\bigcup_{n \in p_2(K \cap A)} (\eta, \omega_1] \times \{n\} = N$ is open and disjoint from $H$. Also, $\bigcup_{m \in p_2(H \cap A)} (\eta, \omega_1] \times \{m\} = M$ is open and disjoint from $K$. Now, consider the clopen (closed-and-open) subspace $Z = \eta + 1 \times \omega + 1$ of $X$ which is normal, being $T_2$-compact. So, the disjoint $Z$-closed subsets $Z \cap H$ and $Z \cap K$ can be separated in $Z$ by, say, $G$ and $L$ with $Z \cap H \subseteq G$ and $Z \cap K \subseteq L$. Now, let $U = M \cup G$ and $V = N \cup L$. Then $U$ and $V$ are disjoint $X$-open subsets with $H \subseteq U$ and $K \subseteq V$. Thus $H$ and $K$ are separated in $X$ which is a contradiction.

**Case 3:** $p_2(K \cap A)$ is bounded and $p_2(H \cap A)$ is bounded. In this case, we must have that either $p_1(H \cap B)$ is bounded or $p_1(K \cap B)$ is bounded since closed unbounded subsets of $\omega_1$ have nonempty intersection and $H$ and $K$ are disjoint. Since either $p_1(H \cap B)$ is bounded or $p_1(K \cap B)$ is bounded, then this case is reduced to either Case 1 or Case 4.

In each case we got a contradiction. Therefore, the Lemma is true.

**Theorem 1.** The Tychonoff Plank $X$ is mildly normal.

**Proof:** Suppose that there exist two disjoint non-empty regularly closed subsets $H$ and $K$ of $X$ which are unseparated. We have that int $H \neq \emptyset \neq$ int $K$. Since any regularly closed set is closed, then, by Lemma 1, assume, without loss of generality, that $p_1(H \cap B)$ is unbounded and $p_2(K \cap A)$ is unbounded.
Claim 1: For each \( n \in p_2(K \cap A) \) and for each \( \alpha < \omega_1 \) there exists \( \beta > \alpha \)
with \( (\beta, n) \in \text{int} K \cap (\omega_1 \times \omega) \).

The statement is clear if \( (\omega_1, n) \in \text{int} K \). If \( (\omega_1, n) \notin \text{int} K \), then for any basic open neighborhood of \( (\omega_1, n) \) which is of the form \( (\alpha, \omega_1] \times \{n\} \), where \( \alpha < \omega_1 \), will meet \( \text{int} K \) because \( (\omega_1, n) \in \overline{\text{int} K} \).

Claim 2: For each \( \gamma \in p_1(H \cap B) \), for each \( \zeta < \gamma \), and for each \( m < \omega \) there exist \( n > m \) and \( \beta \) with \( \zeta < \beta < \gamma \) and \( (\beta, n) \in \text{int} H \cap (\omega_1 \times \omega) \).

The statement is clear if \( (\gamma, \omega) \in \text{int} H \). If \( (\gamma, \omega) \notin \text{int} H \), then for any basic open neighborhood of \( (\gamma, \omega) \) which is of the form \( (\zeta_\gamma, \gamma] \times (m, \omega) \), where \( \zeta_\gamma < \gamma \) and \( m < \omega \), will meet \( \text{int} H \) because \( (\gamma, \omega) \in \overline{\text{int} H} \).

Now, pick \( n_1 \in p_2(K \cap A) \) and \( \alpha_1 < \omega_1 \). By Claim 1, pick \( (\beta_1, n_1) \in \text{int} K \cap (\omega_1 \times \omega) \). Since \( p_1(H \cap B) \) is unbounded, pick \( \gamma_1 \in p_1(H \cap B) \) with \( \beta_1 < \gamma_1 \). Since \( p_2(K \cap A) \) is unbounded, pick \( m_1 \in p_2(K \cap A) \) with \( n_1 < m_1 \). Using Claim 2, pick \( (\alpha_1, k_1) \in \text{int} H \cap (\omega_1 \times \omega) \cap ((\beta_1, \gamma_1] \times (m_1, \omega]) \). We continue by induction.

If for \( l \geq 2 \), \( (\beta_l, n_l), ..., (\beta_l, n_l) \in \text{int} K \cap (\omega_1 \times \omega) \) and \( (\alpha_1, k_1), ..., (\alpha_l, k_l) \in \text{int} H \cap (\omega_1 \times \omega) \) are all picked with \( \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < ... < \beta_l < \alpha_l \) and \( n_1 < k_1 < n_2 < k_2 < ... < n_l < k_l \). Then, since \( p_2(K \cap A) \) is unbounded, pick \( n_{l+1} \in p_2(K \cap A) \). Pick \( (\beta_{l+1}, n_{l+1}) \in \text{int} K \cap (\omega_1 \times \omega) \cap ((\alpha_l, \omega_1] \times \{n_{l+1}\}) \). Since \( p_1(H \cap B) \) is unbounded, pick \( \gamma_{l+1} \in p_1(H \cap B) \) such that \( \beta_{l+1} < \alpha_{l+1} \) and \( m_{l+1} \) \( n_{l+1} < m_{l+1} \). Pick \( (\alpha_{l+1}, k_{l+1}) \in \text{int} H \cap (\omega_1 \times \omega) \cap ((\beta_{l+1}, \gamma_{l+1}] \times (m_{l+1}, \omega]) \). So, by induction, we got two sequences \( \{\beta_i, n_i\} \in \text{int} K \cap (\omega_1 \times \omega) : i \in \mathbb{N} \} \) and \( \{\alpha_i, k_i\} \in \text{int} H \cap (\omega_1 \times \omega) : i \in \mathbb{N} \} \) with \( \beta_i < \alpha_i < \beta_{i+1} < \alpha_{i+1} \) for each \( i \in \mathbb{N} \) and \( n_i < k_i < n_{i+1} < k_{i+1} \) for each \( i \in \mathbb{N} \). Now, the set \( \{\beta_i, \alpha_i : i \in \mathbb{N} \} \) is a countably infinite subset of \( \omega_1 \). Let \( \eta \) be its least upper bound. By our construction, any basic open neighborhood of \( \langle \eta, \omega \rangle \) will meet \( \text{int} H \) and \( \text{int} K \).

Thus \( \langle \eta, \omega \rangle \in \overline{\text{int} H} = H \) and \( \langle \eta, \omega \rangle \in \overline{\text{int} K} = K \). Therefore, \( H \cap K \neq \emptyset \), which is a contradiction. Thus there are no unseparated disjoint regularly closed sets. Thus \( X \) is mildly normal. \( \Box \)

Definition 3 (Singal and Singal, [4]). A topological space \( X \) is called **almost normal** if any two disjoint closed subsets \( A \) and \( B \) of \( X \) one of which is regularly closed can be separated. \( \Box \)

It is clear from the definition that any almost normal space is mildly normal. In [4], Singal and Singal gave a non-regular space which is mildly normal but not almost normal. The next theorem will give a \( T_{\frac{3}{2}} \)-space which is mildly normal but not almost normal.
Theorem 2. The Tychonoff Plank $X$ is not almost normal.

Proof: Let $O = \{2n + 1: n < \omega\}$ and $E = \omega \setminus O$. Let

$$K = \{\langle \omega_1, n\rangle: n \in O\}$$

and

$$H = \left( \bigcup_{m \in E} \{\langle \alpha, m\rangle: \alpha \leq \omega_1, m \in E\} \right) \cup B.$$ 

Now, $\text{int} H = \bigcup_{m \in E} \{\langle \alpha, m\rangle: \alpha \leq \omega_1, m \in E\}$, and hence $\text{int} H = \bigcup_{m \in E} \{\langle \alpha, m\rangle: \alpha \leq \omega_1, m \in E\} = (\bigcup_{m \in E} \{\langle \alpha, m\rangle: \alpha \leq \omega_1, m \in E\}) \cup B = H$. Thus $H$ is regularly closed. It is clear that $K$ is closed and disjoint from $H$. Since $K \subset A$ is infinite and $B \subset H$, then $H$ and $K$ cannot be separated. Thus $X$ is not almost normal. 

REFERENCES


Lutfi N. Kalantan,
King Abdulaziz University, Department of Mathematics,
P.O. Box 114641, Jeddah 21381 – SAUDI ARABIA
E-mail: lkalantan@hotmail.com