BOUTROUX’S METHOD VS. RE-SCALING
Lower estimates for the orders of growth of
the second and fourth Painlevé transcendents

NORBERT STEINMETZ

Abstract: We give a new proof of Shimomura’s sharp lower estimates for the orders of growth of the Painlevé transcendents II and IV: \( \varphi_{II} \geq 3/2 \) and \( \varphi_{IV} \geq 2 \).

1 – Introduction

We are concerned with the transcendental solutions of Painlevé’s second and fourth equation,
\[
W'' = \alpha + zw + 2w^3
\]
and
\[
2ww'' = w^2 + 3w^4 + 8zw^3 + 4(z^2 - \alpha)w^2 + 2\beta,
\]
the second and fourth transcendent. In [Sh1] and [St1] it was shown that any second and fourth Painlevé transcendent \( w \) has order of growth \( \varrho(w) \leq 3 \) and \( \varrho(w) \leq 4 \), respectively. More precisely, if \((p_n)\) denotes the sequence of non-zero poles of \( w \), it was shown in [St1] that, in the respective cases,
\[
\sum_{|p_n| \leq r} |p_n|^{-1} = O(r^2) \quad \text{and} \quad \sum_{|p_n| \leq r} |p_n|^{-2} = O(r^2)
\]
hold. In the other direction, Shimomura [Sh3] recently derived the sharp lower estimates \( \varrho(w) \geq 3/2 \), resp. \( \varrho(w) \geq 2 \). Equality is attained for particular solutions, called Airy- and Hermite–Weber-solutions, respectively. For more details
concerning these functions we refer to [GLS]. For \(2\alpha \in \mathbb{Z}\) proofs of \(g(w) \geq 3/2\) in case (II) can be found in [Sh2] and [St2].

Combining the re-scaling method with some modified Shimomura approach [Sh3] we will be able to give a different proof of Shimomura’s lower estimates, which may be stated as follows:

**Theorem.** Let \(w\) be a transcendental solution of one of the differential equations (1.1) or else (1.2), with sequence \((p_n)\) of non-zero poles. Then, for some \(\kappa = \kappa(w) > 0\) and \(r > r_0\)

\[
\sum_{|p_n| \leq r} |p_n|^{-3/2} \geq \kappa \log r \quad \text{and} \quad \sum_{|p_n| \leq r} |p_n|^{-1/2} \geq \kappa r
\]

or else

\[
\sum_{|p_n| \leq r} |p_n|^{-2} \geq \kappa \log r \quad \text{and} \quad \sum_{|p_n| \leq r} |p_n|^{-1} \geq \kappa r
\]

holds in the respective case.

2 – Two local methods

We start by describing two methods of investigating the Painlevé transcendents locally, and restrict ourselves to equation (1.1).

a) Re-scaling

Let \(w\) be any transcendental solution of (II) and let \((p_n)\) be any sub-sequence of the sequence of poles of \(w\). Then

\[
y_n(\zeta) = p_n^{-1/2} w (p_n + p_n^{-1/2} \zeta)
\]

has the series expansion

\[
y_n(\zeta) = \frac{\epsilon_n}{3} \zeta - \frac{\epsilon_n}{6} \zeta^3 - p_n^{-3/2} 4^{\alpha} + \frac{\epsilon_n}{4} \zeta^3 p_n^{-2} 3^3 + \cdots, \quad \epsilon_n = \pm 1,
\]

about \(\zeta = 0\) and satisfies

\[
y_n''(\zeta) = p_n^{-3/2} \alpha + (p_n^{-3/2} 4 + 1) y_n(\zeta) + 2 y_n^3(\zeta),
\]

where now \(\zeta\) denotes differentiation with respect to \(\zeta\). One of the major results of the re-scaling method developed in [St1, St2] was that the sequence \((h_n p_n^{-2})\) has a
uniform bound only depending on the solution \( w \). Thus choosing a sub-sequence of \((p_n)\), again denoted \((p_n)\), such that \( \epsilon_n = \epsilon \) is constant and \( h_n p_n^{-2} \to h \), we obtain \( y_n(z) \to y_{rs}(z) \) (\( rs \) stands for re-scaled), locally uniformly in \( \mathbb{C} \), where \( y_{rs} \) is the unique solution of

\[
y''_{rs} = y_{rs} + 2 v^3_{rs} \quad \text{with} \quad y_{rs}(3) = \frac{\epsilon}{3} - \frac{\epsilon}{6} \delta + h_3 \delta^2 + \cdots \text{ about } z = 0.
\]

We note that

\[
y''_{rs} = \frac{7}{36} - 10 \epsilon h + y^2_{rs} + y^4_{rs} = c + y^2_{rs} + y^4_{rs},
\]

with \(|c|\) uniformly bounded, independent of the sequence \((p_n)\).

Finally, application of Hurwitz’ Theorem yields the following

**Remark a.** Every pole \( z_0 \) of \( y_{rs} \) is the limit of poles \( z_n \) of \( y_n \); thus \( p'_n = p_n + p_n^{-1/2} z_n \) is a pole of \( w \), and any such sequence \( p'_n \) gives rise to a pole \( z_0 = \lim_{n \to \infty} (p'_n - p_n)p_n^{-1/2} \) of \( y_{rs} \). \( \square \)

**b) Boutroux’s method**

Again let \( w \) be any transcendental solution of (II). The change of variables

\[
\xi = \frac{2}{3} z^{3/2} = \phi(z), \quad \Theta(\xi) = z^{-1/2} w(z),
\]

see Boutroux’s paper [B], leads to the differential equation (where now \( ' \) denotes \( d/d\xi \))

\[
\Theta''(\xi) = 2 \Theta^3(\xi) + \Theta(\xi) - \frac{\Theta'(\xi)}{\xi} + \frac{2\alpha}{3\xi} + \Theta(\xi) + \frac{2}{9\xi^2}.
\]

To be more precise, let \( \mathbb{H} \) be any half-plane with \( 0 \in \partial \mathbb{H} \). Then any branch of \( \psi(\xi) = (\frac{2}{3} \xi)^{2/3} \) maps \( \mathbb{H} \) conformally onto some sector \( S \) of angular width \( 2\pi/3 \), and \( \phi \) will denote the inverse map \( \psi^{-1}: S \to \mathbb{H} \).

If \( p_n \neq 0 \) denotes any pole of \( w \) in the sector \( S \), we obtain for \( v_n(\xi) = \Theta(\phi(p_n) + \xi) \) the differential equation

\[
v''_n(\xi) = 2 v^3_n(\xi) + v_n(\xi) - \frac{v'_n(\xi)}{\phi(p_n) + \xi} + \frac{2\alpha}{3(\phi(p_n) + \xi)} + \frac{v_n(\xi)}{9(\phi(p_n) + \xi)^2}.
\]

If we choose \( p_n \to \infty \) (the same sub-sequence as was chosen above) we obtain in the limit the differential equation

\[
v''_B = 2 v^3_B + v_B,
\]
where \( B \) stands for Boutroux. It is obvious that
\[
\Theta\left(\phi(p_n) + j\right) = \left(\psi(\phi(p_n) + j)\right)^{-1/2} w\left(\psi(\phi(p_n) + j)\right) \sim p_n^{-1/2} w\left(p_n + p_n^{-1/2} j\right)
\]
holds as \( n \to \infty \), and hence the functions \( y_{rs} \) and \( v_B \) agree. This phenomenon was already observed in [St1] for Painlevé’s first equation. The re-scaling method yields the additional information that \( \Theta \) and \( \Theta' \) are uniformly bounded outside the union of disks \( |\xi - \phi(p_n)| < \delta \) about the poles \( \phi(p_n) \) of \( \Theta \); \( \delta > 0 \) is arbitrary. We thus have

**Remark b.** Every pole \( z_0 \) of \( y_B \) is the limit of poles \( z_n \) of \( v_n \); thus \( p'_n = \psi(\phi(p_n) + j) \) is a pole of \( w \), and any such sequence \( p'_n \) gives rise to a pole \( z_0 = \lim_{n \to \infty} (\phi(p'_n) - \phi(p_n)) \) of \( y_B \).

3 – Proof of the Theorem

To start with the proof we need the following Lemma, which in similar form also was proved in [Sh2, Lemma 2.2].

**Lemma.** Let \( \Sigma_c \) denote the (possibly degenerate) period lattice for the differential equation \( y'^2 = y^4 + y^2 + c \), and let \( \Sigma \) be an open sector with vertex at the origin and containing \( \{1, i\} \) (or \( \{-1, i\} \) or \( \{-1, -i\} \) or \( \{1, -i\} \)). Then given \( K > 0 \) there exists \( R > 0 \), such that \( \Sigma \cap \{\omega : |\omega| \leq R\} \cap \Sigma_c \neq \emptyset \) for every \( c \) satisfying \( |c| \leq K \).

**Remark.** For \( c \neq 0, 1/4 \), every non-constant solution of \( y'^2 = y^4 + y^2 + c \) is an elliptic function, closely related to Jacobi’s sinus amplitudinis. If \( \{\omega, \tilde{\omega}\} \) is a suitably chosen basis of the period lattice \( \Sigma \) and if \( \eta \) has a pole at \( j = 0 \), then it has simple poles exactly at \( m\omega + (n + \frac{1}{2})\tilde{\omega} \), \( m, n \in \mathbb{Z} \), see the famous book [HC, p. 215] by Hurwitz and Courant.

**Proof of Lemma:** We have to consider separately the points of degeneration, namely \( c = 0 \), \( c = 1/4 \) and \( c = \infty \). For \( c = 0 \) and \( c = 1/4 \) the non-constant solutions \( \eta \) are simply periodic with primitive periods \( \omega_0 = \pm \pi/\sqrt{2} \) and \( \omega_{1/4} = \pm i\pi \), respectively. Hence, for \( \delta > 0 \) sufficiently small, we have in the respective cases \( |c| < \delta \) and \( |c - 1/4| < \delta \) that, by continuity, one of the periods \( \pm \omega_n \) belong to \( \Sigma \cap \{\omega : |\omega| \leq 4\} \), say.

In case \( c \to \infty \) we set \( u_a(z) = a\eta(az) \) with \( a^4 c = 1 \), to obtain \( u'^2 = u^4 + a^2 u^2 + 1 \), and hence, in the limit \( c \to \infty \), the differential equation \( u'^2 = u^4 + 1 \).
Thus, for $|c|$ large, the period lattice $\mathfrak{L}_c$ is approximately a square lattice with mesh size $\asymp |c|^{-1/4}$. We again note, however, that in our case $|c|$ is uniformly bounded.

In the compact parameter set $\{c: \delta \leq |c| \leq K, |c - 1/4| \geq \delta\}$ each lattice $\mathfrak{L}_c$ has a basis $\{\omega_c, \tilde{\omega}_c\}$ such that $\kappa R \leq |\omega_c| \leq |\tilde{\omega}_c| \leq |\omega_c \pm \tilde{\omega}_c| \leq R$ for some constants $R \geq 4, \kappa > 0$, independent of $c$. The problem now is equivalent to the following: Let $\mathfrak{L}$ be the lattice spanned by 1 and $\tau$ with

$$\text{Im} \tau > 0, \quad -1/2 < \text{Re} \tau \leq 1/2 \quad \text{and} \quad 1 \leq |\tau| \leq M,$$

$M > 1$ some fixed constant, and let $\Sigma$ be any open sector with vertex at the origin and with angular width $\pi/2$. Then we have to show that

$$\Sigma \cap \{1, 1 + \tau, \tau, -1 + \tau, -1, -1 - \tau, -\tau, 1 - \tau\} \neq \emptyset.$$ 

This, however, follows immediately from the fact that the angle between any two consecutive points in the sequence $(1, 1 + \tau, \ldots, 1 - \tau, 1)$ is $< \pi/2$. 

**Proof of the Theorem in case (II):** To fix ideas we consider (the branches of)

$$\psi(\xi) = (\frac{3}{2} \xi)^{2/3}$$

in the half-plane $\mathbb{H}: -\pi/4 < \arg \xi < 3\pi/4$ with $\psi(\mathbb{H}) = S = \{z: -\pi/6 < \arg z < \pi/2\}$ (the other possibilities being $\psi(\mathbb{H}) = e^{2\pi i/3} S$ and $\psi(\mathbb{H}) = e^{4\pi i/3} S$). We also set, for $z_0 \in \psi(\mathbb{H})$, $D(z_0) = \psi(\phi(z_0) + \mathbb{H})$. Then, if $r > 0$ is sufficiently large, it follows from the Lemma and Remarks a. and b. about the distribution of poles, that to any pole $p$ of $w$ in $D(re^{\pi i/6})$ there exists a pole $\phi(p')$ of $\Theta$ in $\phi(p) + \mathbb{H}$ with $|\phi(p') - \phi(p)| \leq 2R$, say. Hence $p' \in D(p) \subset D(re^{\pi i/6})$ is a pole of $w$ satisfying $\frac{3}{2} |p'|^{3/2} - p^{3/2} | \geq \frac{1}{2} |p|^{1/2} |p' - p|$, for $r$ and thus $|p|$ sufficiently large, and this gives $|p' - p| \leq 4 R |p|^{-1/2}$.

Since $D(p') \subset D(p) \subset D(re^{\pi i/6})$, this process may be repeated to obtain a sequence $\tilde{p}_1 = p, \tilde{p}_2 = \tilde{p}_1', \tilde{p}_3 = \tilde{p}_2', \ldots$ of different poles of $w$ such that $|\tilde{p}_{n+1}| \leq |\tilde{p}_n| + O(|\tilde{p}_n|^{-1/2}) = |\tilde{p}_n| (1 + O(|\tilde{p}_n|^{-3/2}))$ as $n \to \infty$. This gives $|\tilde{p}_{n+1} - |\tilde{p}_1| = O\left( \sum_{\nu=1}^{n} |\tilde{p}_\nu|^{-1/2} \right)$ and $\log |\tilde{p}_{n+1}| - \log |\tilde{p}_1| = O\left( \sum_{\nu=1}^{n} |\tilde{p}_\nu|^{-3/2} \right)$ for every $n \in \mathbb{N}$. The assertion of our theorem in case (II) now follows, since the same method applies to the open half-plane $i\mathbb{H}$ with associated sectors $e^{\pi i/3} S$, $-S$ and $e^{5\pi i/3} S$; then the domain $\bigcup_{\nu=0}^{5} D(re^{(2\nu+1)\pi i/6})$ is some punctured neighbourhood of $\infty$. 

The crucial point was to prove that the construction leads to an infinite sequence of poles. It was Shimomura’s paper $[Sh3]$ which inspired me to compare Boutroux’s method with re-scaling and so to overcome this difficulty.
The proof in case (IV) is almost the same, details will be omitted. We just note that, after some simple calculation, the re-scaling process $y_n(z) = p_n^{-1}w(p_n + p_n^{-1}z)$ leads to the differential equation

$$y''_{rs} = y^4 + 4y^3 + 4y^2 + cy,$$

with $|c|$ uniformly bounded. The degenerate cases correspond to the parameters $c = 0, c = 32/27$ and $c = \infty$; by the substitution $u(z) = a\eta(a^3), a^3c = 1$, the latter case again reduces in the limit $c \to \infty$ to $u^2 = u^4 + 1$. One also has to work with (the branches of) $\psi(\xi) = (2\xi)^{1/2}$ in the half-planes $H : -\pi/4 < \arg \xi < 3\pi/4$, $iH$, $-H$ and $-iH$. ■

REFERENCES


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Norbert Steinmetz,
Universität Dortmund, Fachbereich Mathematik,
D-44221 Dortmund – GERMANY
E-mail: stein@math.uni-dortmund.de
www: http://www.mathematik.uni-dortmund.de/steinmetz/