OSCILLATION OF
THIRD-ORDER DIFFERENCE EQUATIONS

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Abstract: In this paper we will study the oscillatory properties of third order
difference equations. By means of the Reccati transformation techniques we will establish
some sufficient conditions which are sufficient for all solutions to be oscillatory or tend
to zero.

1 – Introduction

In recent years, the oscillation theory and the asymptotic behavior of difference
equations and their applications have been and still are receiving intensive
attention. In fact, in the last few years several monographs and hundreds of
research papers have been written, see for example the monographs [1, 2, 4].
Compared to the second order difference equations, the study of third order difference
equations has received considerably less attention in the literature, even
even though such equations arise in the study of Economics, Mathematical Biology,
and other areas of mathematics where discrete models are used (see for example [3]). Some recent results on third order difference equations can be found in
[5-10].

In this paper we shall consider the third order difference equation

\[
\Delta^3 V_n + P_n V_{n+1} = 0, \quad n \geq n_0,
\]

where \( P_n > 0 \) for \( n \geq n_0 \) and \( \Delta \) denotes the forward difference operator \( \Delta V_n = V_{n+1} - V_n \) for any sequence \( \{V_n\} \) of real numbers.

\[\text{Received: January 12, 2003; Revised: March 10, 2003.}\]
By a solution of (1.1) we mean a nontrivial real sequence \( \{V_n\} \) that is defined for \( n \geq n_0 \) and satisfies equation (1.1) for \( n \geq n_0 \). A solution \( \{V_n\} \) of (1.1) is said to be oscillatory, if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

A number of dynamical behavior of solutions of difference equations are possible; here we will only be concerned with conditions which are sufficient for all solutions of (1.1) to be oscillatory or tend to zero as \( n \to \infty \).

2 – Main results

In this section, by using the Reccati transformation techniques we establish some new conditions which are sufficient for all solutions of (1.1) to be oscillatory or tend to zero as \( n \to \infty \).

**Theorem 2.1.** Assume that

\[
\sum_{l=n_3}^{\infty} \left[ \sum_{t=n_4}^{t-1} \sum_{s=n_2}^{t-1} P_t \right] = \infty ,
\]

and there exists a positive sequence \( \{\rho_n\}_{n=n_0}^{\infty} \) such that,

\[
\lim_{n \to \infty} \sup_{n_2} \sum_{l=n_2}^{n} \left[ \rho_l P_l - \frac{(\Delta \rho_l)^2}{4 \rho_l(l-1)} \right] = \infty , \quad \text{for } n_2 > n_1 .
\]

Then every solution \( \{V_n\} \) of Eq.(1.1) oscillates or \( \lim_{n \to \infty} V_n = 0 \).

**Proof:** Let \( \{V_n\} \) be a nonoscillatory solution of (1.1). Without loss of generality we may assume that \( V_n > 0 \) for \( n \geq n_1 \) where \( n_1 \geq n_0 \) is chosen so large. From (1.1) we have \( \Delta^3 V_n \leq 0 \) for \( n \geq n_1 \). Then \( \{V_n\}, \{\Delta V_n\} \) and \( \{\Delta^2 V_n\} \) are monotone and eventually of one sign. We claim \( \Delta^2 V_n > 0 \). Suppose to the contrary that \( \Delta^2 V_n \leq 0 \) for \( n \geq n_2 \) for \( n_2 \geq n_1 \). Since \( \Delta^2 V_n \) is nonincreasing there exists a negative constant \( C \) and \( n_3 \geq n_2 \) such that \( \Delta^2 V_n \leq C \) for \( n \geq n_3 \).

Summing from \( n_3 \) to \( n - 1 \), we obtain

\[
\Delta V_n \leq \Delta V_{n_3} + C(n - 1 - n_3) .
\]

Letting \( n \to \infty \), then \( \Delta V_n \to -\infty \). Thus, there is an integer \( n_4 \geq n_3 \) such that for \( n \geq n_4 \), \( \Delta V_n \leq \Delta V_{n_4} < 0 \). Summing from \( n_4 \) to \( n - 1 \) we obtain

\[
V_n - V_{n_4} \leq C(n - 1 - n_4) ,
\]
this implies that $V_n \to -\infty$ as $n \to \infty$ which is a contradiction with the fact that $V_n$ is positive. Then $\Delta^2 V_n > 0$. Therefore, there are only the following two cases for $n \geq n_1$ sufficiently large:

(I) $V_n > 0, \quad \Delta V_n > 0, \quad \Delta^2 V_n > 0.$

(II) $V_n > 0, \quad \Delta V_n < 0, \quad \Delta^2 V_n > 0.$

First we consider the Case (I): Define $w_n$ by the \textit{Reccati substitution}

\begin{equation}
(2.3) \quad w_n = \frac{\Delta^2 V_n}{V_{n+1}}, \quad n \geq n_1
\end{equation}

we have $w_n > 0$ and

\begin{equation}
\Delta w_n = \Delta^2 V_{n+1} \Delta \left[ \frac{\rho_n}{V_{n+1}} \right] + \frac{\rho_n \Delta^2 V_n}{V_{n+1}},
\end{equation}

this and (1.1), imply that

\begin{equation}
(2.4) \quad \Delta w_n \leq -\rho_n P_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n \Delta^2 V_n \Delta (V_{n+1})}{V_{n+1} V_{n+2}},
\end{equation}

From the Case (I) we have $V_{n+2} \geq V_{n+1}$, then from (2.4) we obtain

\begin{equation}
(2.5) \quad \Delta w_n \leq -\rho_n P_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n \Delta^2 V_{n+1} \Delta (V_{n+1})}{V_{n+2}}.
\end{equation}

Also From the Case (I) and Eq.(1.1) we have

\begin{equation}
(2.6) \quad \Delta V_n = \Delta V_{n_1} + \sum_{s=n_1}^{n-1} \Delta^2 V_s \geq (n - 1 - n_1) \Delta^2 V_n, \quad n \geq n_1 + 1.
\end{equation}

This implies that

\begin{equation}
(2.7) \quad \Delta V_{n+1} \geq (n - n_1) \Delta^2 V_{n+1}, \quad n \geq n_2 = n_1 + 1,
\end{equation}

Substituting from (2.7) in (2.8), we obtain

\begin{equation}
(2.8) \quad \Delta w_n \leq -\rho_n P_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n (n - n_1) (\Delta^2 V_{n+1})^2}{V_{n+2}}.
\end{equation}

From (2.3) and (2.8) we obtain

\begin{equation}
(2.9) \quad \Delta w_n \leq -\rho_n P_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n (n - n_1) \rho_{n+1}^2}{\rho_{n+1}^2} w_{n+1}^2.
\end{equation}
By completing the square we have
\[
\Delta w_n \leq -\rho_n P_n + \frac{(\Delta \rho_n)^2}{4\rho_n(n-n_1)} - \left[ \frac{\sqrt{\rho_n(n-n_1)}}{\rho_{n+1}} w_{n+1} - \frac{\Delta \rho_n}{2\sqrt{\rho_n(n-n_1)}} \right]^2
\]
\[
< - \left[ \rho_n P_n - \frac{(\Delta \rho_n)^2}{4\rho_n(n-n_1)} \right].
\]
Then, we have
\[
(2.10) \quad \Delta w_n < - \left[ \rho_n P_n - \frac{(\Delta \rho_n)^2}{4\rho_n(n-n_1)} \right].
\]
Summing (3.11) from \(n_2\) to \(n\), we obtain

\[
-w_{n_2} < w_{n+1} - w_{n_2} < - \sum_{l=n_2}^{n} \left[ \rho_l P_l - \frac{(\Delta \rho_l)^2}{4\rho_l(l-n_1)} \right],
\]
which yields
\[
(2.11) \quad \sum_{l=n_2}^{n} \left[ \rho_l P_l - \frac{(\Delta \rho_l)^2}{4\rho_l(l-n_1)} \right] < c_1,
\]
for all \(n\), and this is contrary to (2.2). Next we assume that the Case (II) holds. Since \(\{V_n\}\) is positive and decreasing it follows that \(\lim_{n \to \infty} V_n = b \geq 0\).

Now we claim that \(b = 0\). If not then \(V_n \to b > 0\) as \(n \to \infty\), and hence there exists \(n_2 \geq n_1\) such that \(V_{n+1} \geq b\). Therefore from (1.1) we have

\[
(2.12) \quad \Delta^3 V_n + P_n b \leq 0, \quad n \geq n_2,
\]
Define the sequence \(u_n = \Delta^2 V_n\) for \(n \geq n_2\). Then we have

\[
\Delta u_n \leq -P_n b.
\]
Summing the last inequality from \(n_2\) to \(n - 1\), we have

\[
(2.13) \quad u_n \leq u_{n_2} - b \sum_{s=n_2}^{n-1} P_s,
\]
From (2.2), by choosing \(\rho_n = 1\) we have \(\sum_{n=n_0}^{\infty} P_n = \infty\), and then from (2.13) it is possible to choose an integer \(n_3\) sufficiently large such that for all \(n \geq n_3\)

\[
u_n \leq -\frac{b}{2} \sum_{s=n_2}^{n-1} P_s,
\]
and hence
\[ \Delta^2 V_n \leq -\frac{b}{2} \sum_{s=n_2}^{n-1} P_s . \]

Summing the last inequality from \( n_3 \) to \( n - 1 \) we obtain
\[ \Delta V_n \leq \Delta V_{n_3} - \frac{b}{2} \sum_{t=n_3}^{n-1} \left( \sum_{s=n_2}^{t-1} P_s \right) . \]

Since \( \Delta V_n < 0 \) for \( n \geq n_0 \), the last inequality implies that
\[ \Delta V_n \leq -\frac{b}{2} \sum_{t=n_3}^{n-1} \left( \sum_{s=n_2}^{t-1} P_s \right) . \]

Summing from \( n_3 \) to \( n - 1 \) we have
\[ V_n \leq V_{n_3} - \frac{b}{2} \sum_{t=n_3}^{n-1} \left[ \sum_{s=n_2}^{t-1} \sum_{s=n_2}^{t-1} P_s \right] . \]

Condition (2.1) implies that \( V_n \to -\infty \) as \( n \to \infty \) which is a contradiction with the fact that \( V_n \) is positive. Then \( b = 0 \) and this completes the proof.

**Remark 2.1.** From Theorem 2.1, we can obtain different conditions for oscillation of all solutions of (1.1) by different choices of \( \{\rho_n\} \). Let \( \rho_n = n^\lambda \), \( n \geq n_0 \) and \( \lambda \geq 1 \) is a constant. Hence we have the following results.

**Corollary 2.1.** Assume that all the assumptions of Theorem 2.1 hold, except that the condition (2.2) is replaced by
\[ \lim_{n \to \infty} \sup_{n_3} \sum_{s=n_2}^{n} \left[ s^\lambda P_s - \frac{(s + 1)^\lambda - s^\lambda}{4 s^\lambda(s - n_1)} \right] = \infty, \quad \text{for} \quad n_2 > n_1 . \]

Then, every solution \( \{V_n\} \) of Eq.(1.1) oscillates or \( \lim_{n \to \infty} V_n = 0 \).

**Theorem 2.2.** Assume that (2.1) holds. Let \( \{\rho_n\}_{n=n_0}^\infty \) be a positive sequence. Furthermore, we assume that there exists a double sequence \( \{H_{m,n} \colon m \geq n \geq 0\} \) such that

(i) \( H_{m,m} = 0 \) for \( m \geq 0 \),
(ii) \( H_{m,n} > 0 \) for \( m > n \geq 0 \),
(iii) \( \Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \leq 0 \) for \( m \geq n \geq 0 \).
If

\[ \lim_{m \to \infty} \sup_{n=n_2} \frac{1}{H_{m,n_2}} \sum_{n=n_2}^{m-1} \left[ H_{m,n} \rho_n P_n - \frac{(\rho_{n+1})^2}{4\rho_n(n-1)} \left( h_{m,n} - \frac{\Delta \rho_n}{\rho_{n+1} \sqrt{H_{m,n}}} \right)^2 \right] = \infty. \]

for \( n_2 > n_1 \), where

\[ h_{m,n} = -\frac{\Delta_2 H_{m,n}}{\sqrt{H_{m,n}}}, \quad m > n \geq 0. \]

Then, every solution \( \{ V_n \} \) of Eq.(1.1) oscillates or \( \lim_{n \to \infty} V_n = 0 \).

**Proof:** Proceeding as in Theorem 2.1, we assume that Eq.(1.1) has a nonoscillatory solution, say \( V_n > 0 \) for all \( n \geq n_1 \). From the proof of Theorem 2.1 there are two possible cases. First, we consider the case Case (I). Defining again \( \{ w_n \} \) by (2.3), then from Theorem 2.1, we have \( w_n > 0 \) and (2.9) holds. From (2.9) we have for \( n \geq n_2 \)

\[ \rho_n P_n \leq -\Delta w_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n}{(\rho_{n+1})^2} w_{n+1}^2. \]

Therefore, we have

\[ \sum_{n=n_2}^{m-1} H_{m,n} \rho_n P_n \leq -\sum_{n=n_2}^{m-1} H_{m,n} \Delta w_n + \sum_{n=n_2}^{m-1} H_{m,n} \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \sum_{n=n_2}^{m-1} H_{m,n} \frac{\rho_n}{(\rho_{n+1})^2} w_{n+1}^2. \]

which yields after summing by parts

\[ \sum_{n=n_2}^{m-1} H_{m,n} \rho_n P_n \leq H_{m,n_2} w_{n_2} + \sum_{n=n_2}^{m-1} w_{n+1} \Delta_2 H_{m,n} \]

\[ + \sum_{n=n_2}^{m-1} H_{m,n} \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \sum_{n=n_2}^{m-1} H_{m,n} \frac{\rho_n}{(\rho_{n+1})^2} w_{n+1}^2. \]
OSCILLATION OF THIRD-ORDER DIFFERENCE EQUATIONS

hence

\[ \sum_{n=n_2}^{m-1} H_{m,n} \rho_n P_n = \]

\[ = H_{m,n_2} w_{n_2} - \sum_{n=n_2}^{m-1} h_{m,n} \sqrt{H_{m,n}} w_{n+1} + \sum_{n=n_2}^{m-1} H_{m,n} \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} \]

\[ - \sum_{n=n_2}^{m-1} H_{m,n} \left( \frac{\rho_n}{(\rho_{n+1})^2} \right) w_{n+1}^2 \]

\[ = H_{m,n_2} w_{n_2} \]

\[ - \sum_{n=n_2}^{m-1} \left( \frac{\sqrt{H_{m,n}}}{\rho_{n+1}} w_{n+1} + \frac{\rho_{n+1}}{2 \sqrt{H_{m,n} \rho_n}} \left( h_{m,n} \sqrt{H_{m,n}} - \frac{\Delta \rho_n}{\rho_{n+1}} H_{m,n} \right) \right)^2 \]

\[ + \frac{1}{4} \sum_{n=n_2}^{m-1} \left( \frac{\rho_{n+1}}{\rho_n} \right)^2 \left( h_{m,n} - \frac{\Delta \rho_n}{\rho_{n+1}} \sqrt{H_{m,n}} \right)^2 . \]

Then,

\[ \sum_{n=n_2}^{m-1} \left[ H_{m,n} \rho_n P_n - \frac{(\rho_{n+1})^2}{4 \rho_n} \left( h_{m,n} - \frac{\Delta \rho_n}{\rho_{n+1}} \sqrt{H_{m,n}} \right)^2 \right] < H_{m,n_2} w_{n_2} , \]

which implies that

\[ \lim_{m \to \infty} \sup \frac{1}{H_{m,n_2}} \sum_{n=n_2}^{m-1} \left[ H_{m,n} \rho_n P_n - \frac{(\rho_{n+1})^2}{4 \rho_n} \left( h_{m,n} - \frac{\Delta \rho_n}{\rho_{n+1}} \sqrt{H_{m,n}} \right)^2 \right] < w_{n_2} < \infty , \]

which contradicts (2.14). If the Case (II) holds we are then back to the proof of the second case of Theorem 2.1 to prove that \( \lim_{n \to \infty} V_n = 0 \). The proof is complete.

**Remark 2.2.** By choosing the sequence \( \{H_{m,n}\} \) in appropriate ways, we can derive several oscillation criteria for (1.1). For instance, let us consider the double sequence \( \{H_{m,n}\} \) defined by

\[ H_{m,n} = (m - n)^\lambda , \quad \lambda \geq 1, \quad m \geq n \geq 0 , \]

or

\[ H_{m,n} = \left( \log \frac{m + 1}{n + 1} \right)^\lambda , \quad \lambda \geq 1, \quad m \geq n \geq 0 , \]
or

\[ H_{m,n} = (m-n)^{(\lambda)}, \quad \lambda > 2, \ m \geq n \geq 0. \]

where \((m-n)^{(\lambda)} = (m-n)(m-n+1) \cdots (m-n+\lambda-1)\) and

\[ \Delta_2(m-n)^{(\lambda)} = (m-n-1)^{(\lambda)} - (m-n)^{(\lambda)} = -\lambda(m-n)^{(\lambda-1)}. \]

Then \(H_{m,m} = 0\) for \(m \geq 0\) and \(H_{m,n} > 0\) and \(\Delta_2 H_{m,n} \leq 0\) for \(m > n \geq 0\).

Hence we have the following results.

**Corollary 2.2.** Assume that all the assumptions of Theorem 2.2 hold, except that the condition (2.14) is replaced by

\[
\lim_{m \to \infty} \sup \frac{1}{m^\lambda} \sum_{n=2}^{m-1} \left[ (m-n)^\lambda \rho_n P_n - \frac{\rho_{n+1}^2}{4\rho_n(n-n_1)} \left( \lambda(m-n)^{\frac{\lambda-2}{2}} - \frac{\Delta \rho_n}{\rho_{n+1} \sqrt{(m-n)\lambda}} \right)^2 \right] = \infty.
\]

Then, every solution \(\{V_n\}\) of Eq.(1.1) oscillates or \(\lim_{n \to \infty} V_n = 0\).

**Corollary 2.3.** Assume that all the assumptions of Theorem 2.2 hold, except that the condition (2.14) is replaced by

\[
\lim_{m \to \infty} \sup \frac{1}{(\log(m+1))^{\lambda}} \sum_{n=2}^{m-1} \left[ \left( \log \frac{m+1}{n+1} \right)^\lambda \rho_n P_n - \frac{\rho_{n+1}^2}{4\rho_n(n-n_1)} \left( \lambda \left( \log \frac{m+1}{n+1} \right)^{\frac{\lambda-2}{2}} - \frac{\Delta \rho_n}{\rho_{n+1} \sqrt{\left( \log \frac{m+1}{n+1} \right)^\lambda}} \right)^2 \right] = \infty.
\]

Then, every solution \(\{V_n\}\) of Eq.(1.1) oscillates or \(\lim_{n \to \infty} V_n = 0\).

**Corollary 2.4.** Assume that all the assumptions of Theorem 2.2 hold, except that the condition (2.14) is replaced by

\[
\lim_{m \to \infty} \sup \frac{1}{m^{\lambda}} \sum_{n=2}^{m-1} \left( m-n \right)^{(\lambda)} \left[ \rho_n P_n - \frac{\rho_{n+1}^2}{4\rho_n(n-n_1)} \left( \frac{\lambda}{m-n+\lambda-1} - \frac{\Delta \rho_n}{\rho_{n+1}} \right)^2 \right] = \infty.
\]

Then, every solution \(\{V_n\}\) of Eq.(1.1) oscillates or \(\lim_{n \to \infty} V_n = 0\).
Remark 2.3. Our results can be extended to nonlinear difference equations of the form

\[ \Delta^3 V_n + P_n f(V_{n+1}) = 0, \quad n \geq n_0. \]

where \( f: \mathbb{R} \to \mathbb{R} \) is continuous such that \( uf(u) > 0 \) for \( u \neq 0 \) and \( f(u)/u \geq K > 0 \) except that the term \( P_n \) is replaced by \( KP_n \).

ACKNOWLEDGEMENT – The author would like to thank the referee for his (her) helpful comments.

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