MULTIPLE SEMICLASSICAL SOLUTIONS
OF THE SCHRÖDINGER EQUATION
INVOLVING A CRITICAL SOBOLEV EXponent

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Abstract: We prove the existence of multiple solutions of the Schrödinger equation involving a critical Sobolev exponent. We use the Lusternik–Schnirelman theory of critical points.

1 – Introduction

The main purpose of this work is to investigate the existence of multiple solutions of the Schrödinger equation

\[(1_\epsilon) \quad -\epsilon^2 \Delta u + a(x) u = |u|^{2^* - 2} u \quad \text{in } \mathbb{R}^N\]

for \(\epsilon > 0\) small, where \(2^* = \frac{2N}{N-2}, \ N \geq 3\).

Solutions of equation \((1_\epsilon)\) corresponding to a small parameter \(\epsilon > 0\) are referred to in the existing literature as semiclassical solutions [1], [2], [11], [13], [14], [15], [16]. Problem \((1_\epsilon)\) arises in the search for standing waves for the nonlinear Schrödinger equation

\[i \ h \ \frac{\partial \psi}{\partial t} = -h^2 \Delta \psi + U(x) \psi - |\psi|^{p-2} \psi \quad \text{in } \mathbb{R}^N ,\]

where \(h\) is the Planck constant, \(p > 2\) if \(N = 1, 2\) and \(2 < p \leq 2^*\) if \(N \geq 3\). Standing waves of this equation are solutions of the form \(\psi(t, x) = \exp(-i \lambda h^{-1} t) \ u(x)\),

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where \( u \) is a real-valued function satisfying (1\(_\epsilon\)) with \( a(x) = U(x) + \lambda \) and \( h = e^2 \). Obviously, the equation (1\(_\epsilon\)) corresponds to the case \( p = 2^* \). The first result on the existence of semiclassical solutions was obtained by Floer–Weinstein in [11] via the Lyapunov–Schmidt method in the case \( N = 1 \). This result was extended by Oh [14], [15] to higher dimensions, in the subcritical case \( 2 < p < 2^* \). Some related results can be found in the papers [19], [20], [9] and [11]. It is well known that the existence of multiple solutions for the Dirichlet problem for (1) on bounded domains depends on the topology of this domain (see for example [4], [6]). In the case of problem (1\(_\epsilon\)) a similar role is played by the graph topology of coefficient \( a \). This phenomenon also occurs for the Dirichlet problem on bounded domains [6]. The effect of the graph topology of the coefficients on the existence of multiple solutions in the subcritical case was investigated in the papers [9] and [13] and in [18] for the Dirichlet problem in bounded domains. The aim of this paper is to relate the number of solutions of problem (1\(_\epsilon\)) with \( \text{cat}\ a^{-1}(0) \). It is well known that if \( a(x) = \text{Constant} \neq 0 \), problem (1\(_\epsilon\)) has no solution by the Pohozaev identity. A similar situation occurs also for the Dirichlet problem on bounded starshaped domains if \( a(x) = \text{Constant} \geq 0 \). However, in the case \( a(x) \neq \text{Constant} \) there are existence results for (1\(_\epsilon\)), with \( \epsilon = 1 \), (see for example [3]) and for the Dirichlet problem on bounded domains [18] under some structural assumptions on \( a(x) \). For further bibliographical references on the effect of the coefficient \( a(x) \) on the existence and nonexistence of solutions, we refer to the papers [3] and [18].

Throughout this paper we use standard notation and terminology. In a given Banach space \( X \), we denote by “→” a strong convergence and by “→∗” a weak convergence. Let \( F \in C^1(X, \mathbb{R}) \). A sequence \( \{u_n\} \) is said to be the Palais–Smale sequence for \( F \) at a level \( c \) ((PS)\(_c\)-sequence for short) if \( F(u_n) \to c \) and \( F'(u_n) \to 0 \) in \( X^* \) as \( n \to \infty \). We say that \( F \) satisfies the Palais–Smale condition at level \( c \) ((PS)\(_c\) condition for short) if every (PS)\(_c\) sequence is relatively compact in \( X \).

By \( D^{1,2}(\mathbb{R}^N) \) we denote the Sobolev space obtained as the closure of \( C_0^\infty(\mathbb{R}^N) \) with respect to the norm

\[
\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx .
\]

By \( B(y, R) \) we always denote an open ball in \( \mathbb{R}^N \) centered at \( y \) of radius \( R \).
2 – Preliminaries

Throughout this paper we assume that the potential \(a(x)\) satisfies the following two conditions:

\((A_1)\) \(a(x) \geq 0\) on \(\mathbb{R}^N\) and the set \(M = \{x \in \mathbb{R}^N; a(x) = 0\}\) is nonempty and bounded.

\((A_2)\) There exist two constants \(p_1 < \frac{N}{2}\) and \(p_2 > \frac{N}{2}\) (with \(p_2 < 3\) if \(N = 3\)) such that \(a \in L^p(\mathbb{R}^N)\) for each \(p \in [p_1, p_2]\).

Benci–Cerami [3] established the existence of a positive solution of the equation \((1_\epsilon)\), with \(\epsilon = 1\), and with \(a\) satisfying \((A_2)\) and \(\|a\|_{\frac{N}{2}} \leq S(\frac{N}{2} - 1)\). Here \(S\) denotes the best Sobolev constant for a continuous embedding of \(D^{1,2}(\mathbb{R}^N)\) into \(L^{2^*}(\mathbb{R}^N)\), that is,

\[
S = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx; \|u\|_{2^*} = 1 \right\}.
\]

In this paper we examine the effect of topology of the set \(M\) on the number of solutions of \((1_\epsilon)\).

We set for \(\delta > 0\) small

\[
M_\delta = \{x \in \mathbb{R}^N; \text{dist}(x, M) \leq \delta\}
\]

and

\[
\Sigma = \left\{ u \in D^{1,2}(\mathbb{R}^N); \int_{\mathbb{R}^N} |u(x)|^{2^*} \, dx = 1 \right\}.
\]

It is well known that the positive solutions which are radially symmetric about some point in \(\mathbb{R}^N\) of the equation

\[
(2) \quad -\Delta u = |u|^{2^*-2} u \quad \text{in} \quad \mathbb{R}^N
\]

have form

\[
U_{\lambda,y}(x) = \left[\frac{N(N-2)\lambda}{(\lambda + |x-y|^2)^{\frac{N+2}{2}}}\right]^\frac{N-2}{4}, \quad \lambda > 0, \ y \in \mathbb{R}^N,
\]

with

\[
\|U_{\lambda,y}\|_{2^*} = S^{\frac{N-2}{4}} \quad \text{and} \quad \|\nabla U_{\lambda,y}\|^2 = S^\frac{N}{2}.
\]

Let \(\tilde{U}_{\lambda,y}(x) = S^{-\frac{N-2}{4}} U_{\lambda,y}(x)\). Then

\[
\|\tilde{U}_{\lambda,y}\|_{2^*} = 1 \quad \text{and} \quad \|\nabla \tilde{U}_{\lambda,y}\|_2 = S.
\]
We define the following functionals on $D^{1,2}(\mathbb{R}^N)$:

\[
J^2(u) = \frac{\epsilon^2}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + a(x)u^2) \, dx ,
\]

\[
I^2(u) = \frac{1}{2} \int_{\mathbb{R}^N} (\epsilon^2 |\nabla u|^2 + a(x)u^2) \, dx - \frac{1}{2s} \int_{\mathbb{R}^N} |u|^2^s \, dx
\]

and

\[
I_\epsilon^\infty(u) = \frac{\epsilon^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1}{2s} \int_{\mathbb{R}^N} |u|^2^s \, dx .
\]

To determine the energy levels of the functional $I_\epsilon$ for which the Palais–Smale condition holds, we need the following result due to Benci–Cerami [3].

**Theorem 1.** Let $\{u_n\}$ be a $(PS)_\epsilon$-sequence for the functional $I_\epsilon$. Then there exist a number $k \in \mathbb{N}$, $k$ sequences of points $\{y^j_n\} \subset \mathbb{R}^N$, $j = 1, ..., k$, $k$ sequences of positive numbers $\{\sigma^j_n\}$, $j = 1, ..., k$ and $k + 1$ sequences of functions $\{u^j_n\} \subset D^{1,2}(\mathbb{R}^N)$, $j = 0, 1, ..., k$, such that, up to a subsequence,

\[
u_n(x) = u^0_n(x) + \sum_{j=1}^k \frac{1}{\sigma^j_n} u^j_n \left( \frac{x - y^j_n}{\sigma^j_n} \right),
\]

(3)

\[
u^j_n(x) \rightharpoonup u^j(x) \quad \text{in} \quad D^{1,2}(\mathbb{R}^N), \quad j = 0, ..., k,
\]

(4)

as $n \to \infty$, where $u^0$ is a solution of equation $(1_\epsilon)$, $u^j$, $j = 1, ..., k$ are solutions of the equation

\[-\epsilon^2 \Delta u = |u|^{2^* - 2} u \quad \text{in} \quad \mathbb{R}^N ,
\]

(5)

and if $y^j_n \to \bar{y}^j$ as $n \to \infty$, then either $\sigma^j_n \to \infty$ or $\sigma^j_n \to 0$ as $n \to \infty$. Furthermore, we have

\[\|u_n\|^2 \rightharpoonup \sum_{j=0}^k \|u^j\|^2\]

and

\[I_\epsilon(u_n) \rightharpoonup I_\epsilon(u^0) + \sum_{j=1}^k I_\epsilon^\infty(u^j)\]

as $n \to \infty$. 

Since for every nontrivial solution $u$ of $(1_\epsilon)$, $I_\epsilon(u) > \frac{\epsilon N}{N} S^N$, for every positive solution $u$ of $(2)$, $I_\epsilon^\infty(u) > \frac{\epsilon N}{N} S^N$ and for every solution $u$ of $(5)$ which changes sign we have $I_\epsilon^\infty(u) \geq \frac{2\epsilon N}{N} S^N$, we deduce from Theorem 1:
Corollary 1. Let \( \{u_n\} \subset D^{1,2}(\mathbb{R}^N) \) be a \((PS)_c\)-sequence for \( I_\epsilon \) with \( \frac{\epsilon^N}{N} S_{\frac{N}{2}}^N < c < \frac{2\epsilon^N}{N} S_{\frac{N}{2}}^N \). Then \( \{u_n\} \) is relatively compact in \( D^{1,2}(\mathbb{R}^N) \).

Corollary 2. The functional \( J_\epsilon|_\Sigma \) satisfies the \((PS)_c\)-condition for \( c \in (\epsilon^2 S, 2\frac{\pi}{\epsilon} \epsilon^2 S) \).

The proof of the following lemma can be found in [3] (see formulae (3.7), (3.9) and (3.19) there).

Lemma 1. We have

\[
\begin{align*}
\lim_{\lambda \to 0} \int_{\mathbb{R}^N} a(x) \bar{U}_{\lambda,y}(x)^2 \, dx &= 0 \quad \text{for every } y \in \mathbb{R}^N, \\
\lim_{\lambda \to \infty} \int_{\mathbb{R}^N} a(x) \bar{U}_{\lambda,y}(x)^2 \, dx &= 0 \quad \text{for every } y \in \mathbb{R}^N, \\
\text{and} \\
\lim_{|y| \to \infty} \int_{\mathbb{R}^N} a(x) \bar{U}_{\lambda,y}(x)^2 \, dx &= 0 \quad \text{for every } \lambda > 0.
\end{align*}
\]

We choose \( \rho > 0 \) such that \( M_\delta \subset B(0, \frac{\rho}{2}) \), \( \rho = \rho(\delta) \). Let

\[
\chi(x) = \begin{cases} 
  x & \text{for } |x| \leq \rho, \\
  \frac{\rho x}{|x|} & \text{for } |x| \geq \rho.
\end{cases}
\]

We define a “barycenter” \( \beta: \Sigma \to \mathbb{R}^N \) by

\[
\beta(u) = \int_{\mathbb{R}^N} \chi(x)|u(x)|^{2^*} \, dx
\]

and set

\[
\gamma(u) = \int_{\mathbb{R}^N} |\chi(x) - \beta(u)| |u(x)|^{2^*} \, dx.
\]

The functional \( \gamma \) measures the concentration of a function \( u \) near its barycenter.

With the aid of \( \bar{U}_{\lambda,y} \) we define a mapping \( \Phi_{\lambda,y}: \mathbb{R}^N \to \Sigma \) by \( \Phi_{\lambda,y}(\cdot) = \bar{U}_{\lambda,y}(\cdot) \).

We note that

\[
\beta(\Phi_{\lambda,y}) = \int_{\mathbb{R}^N} \chi(x) \Phi_{\lambda,y}(x)^2 \, dx
\]

\[
= y + \int_{\mathbb{R}^N} \left( \chi(\lambda z + y) - y \right) \bar{U}_{1,0}(z)^2 \, dz
\]

\[
= y + o(1)
\]

(6)
as $\lambda \to 0$. Let

$$
V = V(\lambda_1, \lambda_2, \rho) = \left\{(y, \lambda) \in \mathbb{R}^N \times \mathbb{R}; \ |y| < \frac{\rho}{2}, \ \lambda_1 < \lambda < \lambda_2 \right\}.
$$

It follows from Lemma 1 that for every $\varepsilon > 0$ there exist $\lambda_1 = \lambda_1(\varepsilon)$ and $\lambda_2 = \lambda_2(\varepsilon)$, with $\lambda_1 < \lambda_2$, such that

$$
\sup \left\{ \varepsilon^2 \int_{\mathbb{R}^N} |\nabla \Phi_{\lambda,y}|^2 \, dx + \int_{\mathbb{R}^N} a(x) \Phi_{\lambda,y}^2 \, dx; \ (y, \lambda) \in V \right\} < \varepsilon^2 (S + h(\varepsilon)),
$$

where $h(\varepsilon) \to 0$ as $\varepsilon \to 0$.

To examine the behaviour of $\gamma \circ \Phi_{\lambda,y}$ as $\lambda \to 0$ we need the following estimate.

**Lemma 2.** Let $0 < 2\varepsilon < \rho$ and $x \in B(y, \varepsilon)$. Then

$$
|\chi(x) - \chi(y)| \leq 2 |x - y| + 2 \varepsilon.
$$

**Proof:** We distinguish three cases: (i) $|y| \geq \rho + \varepsilon$, (ii) $|y| \leq \rho - \varepsilon$ and (iii) $\rho - \varepsilon \leq |y| \leq \rho + \varepsilon$.

**Case (i).** Since $|x| \geq |y| - |x-y| \geq \rho$, we have

$$
|\chi(x) - \chi(y)| = \rho \left| \frac{x}{|x|} - \frac{y}{|y|} \right| = \rho \left| \frac{|x||y| - |y||x|}{|x||y|} \right| = \rho \left| \frac{|x||y| - |y||y| + |y||y| - |y||x|}{|x||y|} \right| \leq \\
\leq \rho \left| \frac{|x - y| + |y| - |x||y|}{|x||y|} \right| \leq 2 |x - y|.
$$

**Case (ii).** We have $|x| \leq |x-y| + |y| \leq \rho - \varepsilon + \varepsilon = \rho$ and $|\chi(x) - \chi(y)| = |x - y|$.

**Case (iii).** In this case $\rho - 2\varepsilon \leq |x| \leq \rho + 2\varepsilon$. If $|x| \leq \rho$ and $|y| \leq \rho$, then $|\chi(x) - \chi(y)| = |x - y|$. If $|x| \geq \rho$ and $|y| \geq \rho$, we show as in the case (i) that $|\chi(x) - \chi(y)| \leq 2 |x - y|$. If $|x| \leq \rho$ and $|y| \geq \rho$, then

$$
|\chi(x) - \chi(y)| = \left| x - \rho \frac{y}{|y|} \right| = \left| \frac{|x||y| - \rho y}{|y|} \right| \leq \left| \frac{|x||y| - |y||y| + |y||y| - \rho y}{|y|} \right| \leq \\
\leq |x - y| + |y| - \rho \leq |x - y| + \varepsilon.
$$

Finally, if $|x| \geq \rho$ and $|y| \leq \rho$, then

$$
|\chi(x) - \chi(y)| = \left| \rho \frac{x}{|x|} - y \right| = \left| \rho \frac{x - y}{|x|} \right| \leq \left| \frac{|x||y| - \rho y}{|y|} \right| \leq \\
\leq \rho \left| \frac{|x - y|}{|x|} + \frac{|y|}{|x|} (|x| - \rho) \right| \leq |x - y| + 2 \varepsilon.
$$
Lemma 3. We have \( \lim_{\lambda \to 0} \gamma \circ \Phi_{\lambda, y} = 0 \) uniformly for \( |y| \leq \frac{\rho}{2} \).

Proof: Let \( 0 < 2 \varepsilon < \rho \). We commence by observing that

\[
\int_{\mathbb{R}^N - B(0, \varepsilon)} \Phi_{\lambda, 0}^2(x) \, dx = C_N \int_{\mathbb{R}^N - B(0, \varepsilon)} \frac{\lambda N}{\lambda + |x|^2} \, dx = C_N \int_{|x| \geq \frac{\rho}{2N}} \frac{1}{1 + |x|^2} \, dx \to 0
\]

as \( \lambda \to 0 \). We write

\[
\int_{\mathbb{R}^N} \left| \chi(x) - \beta \circ \Phi_{\lambda, y} \right| \Phi_{\lambda, y}(x)^2^* \, dx = \int_{B(y, \varepsilon)} \left| \chi(x) - \beta \circ \Phi_{\lambda, y} \right| \Phi_{\lambda, y}(x)^2^* \, dx + \int_{\mathbb{R}^N - B(y, \varepsilon)} \left| \chi(x) - \beta \circ \Phi_{\lambda, y} \right| \Phi_{\lambda, y}(x)^2^* \, dx.
\]

We deduce from (8) that

\[
\lim_{\lambda \to 0} \int_{\mathbb{R}^N - B(y, \varepsilon)} \left| \chi(x) - \beta \circ \Phi_{\lambda, y} \right| \Phi_{\lambda, y}(x)^2^* \, dx = 0.
\]

The integral over \( B(y, \varepsilon) \) can be estimated using Lemma 2 and (6) as follows

\[
\int_{B(y, \varepsilon)} \left| \chi(x) - \beta \circ \Phi_{\lambda, y} \right| \Phi_{\lambda, y}(x)^2^* \, dx \leq \int_{B(y, \varepsilon)} \left| \chi(x) - \chi(y) \right| \Phi_{\lambda, y}(x)^2^* \, dx + \int_{B(y, \varepsilon)} \left| \chi(y) - \beta \circ \Phi_{\lambda, y} \right| \Phi_{\lambda, y}(x)^2^* \, dx
\]

\[
\leq 2 \int_{B(y, \varepsilon)} |x - y| \Phi_{\lambda, y}(x)^2^* \, dx + 2 \varepsilon S + o(1).
\]

Since \( \varepsilon > 0 \) is arbitrary, \( \lim_{\lambda \to 0} \gamma(\Phi_{\lambda, y}) = 0 \). Due to the compactness of \( \{y: |y| \leq \frac{\rho}{2}\} \) this convergence can be made uniform on this set.

We now define a set \( \Sigma_\varepsilon \subset \Sigma \) by

\[
\Sigma_\varepsilon = \left\{ u \in \Sigma; \ S < \varepsilon^2 J_\varepsilon(u) < S + h(\varepsilon), \ (\beta(u), \gamma(u)) \in V \right\},
\]

where \( V \) has been chosen so that (7) holds. According to Lemma 3 we can modify \( \lambda_1(\varepsilon) \) and \( \lambda_2(\varepsilon) \) so that \( \Sigma_\varepsilon \neq \emptyset \) for each \( \varepsilon > 0 \) small.

Proposition 3. We have

\[
\limsup_{\varepsilon \to 0} \inf_{u \in \Sigma_\varepsilon} [\beta(u) - \beta(\Phi_{\lambda, y})] = 0.
\]
Proof: Let \( \{ \epsilon_n \} \) be a sequence of positive numbers such that \( \epsilon_n \to 0 \). For every \( n \) there exists \( u_n \in \Sigma \epsilon_n \) such that

\[
\inf_{y \in M \delta} \left[ \beta(u_n) - \beta(\Phi_{\epsilon_n,y}) \right] = \sup_{u \in \Sigma \epsilon_n} \inf_{y \in M \delta} \left[ \beta(u) - \beta(\Phi_{\epsilon_n,y}) \right] + o(1) .
\]

In order to prove (9) it is sufficient to find a sequence \( \{ y_n \} \subset M \delta \) such that

\[
\lim_{n \to \infty} \left[ \beta(u_n) - \beta(\Phi_{\epsilon_n,y_n}) \right] = 0 .
\]

Since \( \{ u_n \} \subset \Sigma \epsilon_n \) we have

\[
\epsilon_n^2 S \leq \epsilon_n^2 \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \leq \epsilon_n^2 \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + \int_{\mathbb{R}^N} a(x) u_n^2 \, dx \leq \epsilon_n^2 (S + h(\epsilon_n)) .
\]

Hence

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx = S \quad \text{and} \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} a(x) u_n^2 \, dx = 0 .
\]

It then follows from Corollary 2.11 in [3] that there exist a sequence of points \( \{ y_n \} \subset \mathbb{R}^N \), a sequence \( \{ \delta_n \} \subset (0, \infty) \) and a sequence of functions \( \{ w_n \} \subset D^{1,2}(\mathbb{R}^N) \) such that \( w_n \to 0 \) as \( n \to \infty \) and

\[
u_n(x) = w_n(x) + \Phi_{\delta_n,y_n}(x) \quad \text{on} \quad \mathbb{R}^N .
\]

We claim that (i) \( \delta_n \to 0 \) and (ii) \( \{ y_n \} \) is bounded. We begin by showing that \( \{ \delta_n \} \) is bounded. In the contrary case we may assume that \( \delta_n \to \infty \) as \( n \to \infty \). Since \( w_n \to 0 \) in \( D^{1,2}(\mathbb{R}^N) \), we have

\[
\beta(u_n) = \beta(\Phi_{\delta_n,y_n}) + o(1) .
\]

Indeed, (12) follows from the following relation

\[
\begin{align*}
\beta(u_n) &= \int_{\mathbb{R}^N} \chi(x) |u_n|^{2^*} \, dx \\
&= \int_{\mathbb{R}^N} \chi(x) |w_n + \Phi_{\delta_n,y_n}|^{2^*} \, dx \\
&= \int_{\mathbb{R}^N} \chi(x) \Phi_{\delta_n,y_n}^{2^*} \, dx + O \left( \int_{\mathbb{R}^N} |w_n|^{2^* - 1} \Phi_{\delta_n,y_n} \, dx \right) \\
&= \int_{\mathbb{R}^N} \chi(x) \Phi_{\delta_n,y_n}^{2^*} \, dx + O \left( \| w_n \|^{2^* - 1} \| \Phi_{\delta_n,y_n} \|_2^* \right) \\
&= \int_{\mathbb{R}^N} \chi(x) \Phi_{\delta_n,y_n}^{2^*} \, dx + o(1) .
\end{align*}
\]
Therefore we may assume that

$$\beta(\Phi_{\delta_n,y_n}) \subset B\left(0, \frac{\rho}{2}\right).$$

We now observe that for each $R > 0$ we have

$$\lim_{n \to \infty} \int_{B(0,R)} \Phi^2 \delta_n,y_n \, dx = 0,$$

since $\lim_{n \to \infty} \delta_n = \infty$. Using this and (13) we can write the following inequalities

$$\gamma \circ \Phi_{\delta_n,y_n} = \int_{\mathbb{R}^N} |\chi(x) - \beta \circ \Phi_{\delta_n,y_n} | \Phi_{\delta_n,y_n}(x)^{2^*} \, dx$$

$$\geq \int_{\mathbb{R}^N} |\chi(x) | \Phi_{\delta_n,y_n}(x)^{2^*} \, dx - |\beta \circ \Phi_{\delta_n,y_n} |$$

$$\geq \int_{\mathbb{R}^N} |\chi(x) | \Phi_{\delta_n,y_n}(x)^{2^*} \, dx - \frac{\rho}{2}$$

(14)

$$\geq \rho \int_{\mathbb{R}^N - B(0,\rho)} \Phi^2 \delta_n,y_n \, dx - \frac{\rho}{2} + o(1)$$

$$= \rho \int_{\mathbb{R}^N} \Phi^2 \delta_n,y_n \, dx - \frac{\rho}{2} + o(1)$$

$$= \frac{\rho}{2} + o(1).$$

On the other hand we have

$$\gamma(u_n) = \gamma(\Phi_{\delta_n,y_n}) + o(1),$$

because $w_n \to 0$ in $D^{1,2}(\mathbb{R}^N)$. Since $u_n \in \Sigma_{\epsilon_n}$ we have that

$$\lambda_1(\epsilon_n) < \gamma(u_n) < \lambda_2(\epsilon_n)$$

with $\lambda_i(\epsilon_n) \to 0$, $i = 1, 2$, as $\epsilon_n \to \infty$. This contradicts the estimate (14) and therefore $\{\delta_n\}$ is bounded. It remains to show that $\delta_n \to 0$. In the contrary case we may assume that $\delta_n \to \delta > 0$ as $n \to \infty$. Then we must have that $|y_n| \to \infty$ as $n \to \infty$, since otherwise $\Phi_{\delta_n,y_n}$ would converge strongly in $D^{1,2}(\mathbb{R}^N)$ and so would $u_n$. Consequently $J_e$ subject to the constraint $\Sigma$ would have minimizer which is impossible by Proposition 2.2 in [3]. We now observe that for every $R > 0$, the fact that $\lim_{n \to \infty} |y_n| = \infty$, implies that $\lim_{n \to \infty} \int_{B(0,R)} \Phi^2 \delta_n,y_n \, dx = 0$. Consequently one can easily show that the estimate (14) must be valid giving the contradiction with the fact that $u_n$ satisfies (15). The proof of the claim (ii) is similar and it is omitted. We now choose subsequences of $\{\delta_n\}$ and $\{\epsilon_n\}$ so that
\[ \frac{\delta_n}{\epsilon_n} = o(1) \text{ as } n \to \infty. \] So we may replace \( \delta_n \) by \( \epsilon_n \). The new sequence \( \{\epsilon_n\} \) is relabelled again by \( \{\epsilon_n\} \). Suppose that \( y_n \to \bar{y} \). Let
\[ v_n(x) = \frac{\epsilon_n}{\epsilon_n} u_n(\epsilon_n x + y_n). \]
Then \( v_n \to U_{1,0} \) in \( D^{1,2}(\mathbb{R}^N) \). Since \( \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \to S \) and
\[
\epsilon_n^2 S < \epsilon_n^2 \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + \int_{\mathbb{R}^N} a(x) u_n^2 \, dx
\]
\[
\epsilon_n^2 \int_{\mathbb{R}^N} \left( |\nabla v_n|^2 + a(\epsilon_n x + y_n) v_n^2 \right) \, dx
\]
\[ < \epsilon_n^2 (S + h(\epsilon_n)), \]
we deduce that
\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} a(\epsilon_n x + y_n) v_n^2 \, dx = 0. \]
This implies that \( \int_{\mathbb{R}^N} a(\bar{y}) U_{1,0} \, dx = 0 \) and so \( a(\bar{y}) = 0 \). This means that \( \bar{y} \in M \).
Therefore \( y_n \in M_\delta \) for large \( n \). The relation (10) follows from the fact that \( w_n \to 0 \) in \( D^{1,2}(\mathbb{R}^N) \).

3 – Main result

We are now in a position to formulate our main result on the existence of multiple solutions in terms of \( \text{cat}_{M_\delta} M \).

**Theorem 2.** For small \( \epsilon > 0 \) the problem (1\( \epsilon \)) has \( \text{cat}_{M_\delta} M \) solutions.

**Proof:** We fix an \( \epsilon > 0 \) small. Then \( \Phi_{\lambda,\beta}: [\lambda_1, \lambda_2] \times M \to \Sigma_\epsilon \) and by virtue of (6) and Proposition 3, \( \beta(\Sigma_\epsilon) \subset M_\delta \). Therefore \( \beta \circ \Phi_{\lambda,\beta}: [\lambda_1, \lambda_2] \times M \to [\lambda_1, \lambda_2] \times M_\delta \) and it is easy to check that \( \beta \circ \Phi_{\lambda,\beta}: [\lambda_1, \lambda_2] \times M \to [\lambda_1, \lambda_2] \times M_\delta \) is homotopic to the inclusion map \( \text{id}: [\lambda_1, \lambda_2] \times M \to [\lambda_1, \lambda_2] \times M_\delta \). The functional \( J_\epsilon \) satisfies the \( (PS)_c \)-condition for \( c \in (\epsilon^2 S, \epsilon^2 (S + h(\epsilon))) \). Hence by the Lusternik–Schnirelman theory of critical points (see [3], [4], [5])
\[
\text{cat}(\Sigma_\epsilon) \geq \text{cat}_{[\lambda_1, \lambda_2] \times M_\delta} ([\lambda_1, \lambda_2] \times M) = \text{cat}_{M_\delta} M. \]

**Remark.** Using the argument of Lemma 2.7 in [18] one can show that solutions obtained in Theorem 2 are positive. \( \square \)
In the next result we show that solutions $u_\epsilon$ obtained in Theorem 2 concentrate on $M$ as $\epsilon \to 0$.

**Theorem 3.** Let $\{u_\epsilon\}$ be solutions from Theorem 2. Then $u_\epsilon \to \tilde{U}_{0,\tilde{y}}$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ as $\epsilon \to 0$ and $\tilde{y} \in M$.

**Proof:** It follows from (11), that
\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^N} |\nabla u_\epsilon|^2 \, dx = S \quad \text{and} \quad \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} a(x) u_\epsilon^2 \, dx = 0.
\]
Thus $u_\epsilon = w_\epsilon + \Phi_{\delta_\epsilon,y_\epsilon}$. As in Proposition 3 we show that $\delta_\epsilon \to 0$ and $y_\epsilon \to \tilde{y} \in M$ as $\epsilon \to 0$. □

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