SKEW SEMI-INVARIANT SUBMANIFOLDS
OF A LOCALLY PRODUCT MANIFOLD

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Abstract: In this paper, we defined and studied a new class of submanifolds of a locally Riemannian product manifold, i.e., skew semi-invariant submanifolds. We give two sufficient conditions for submanifolds to be skew semi-invariant submanifolds. Moreover, we discussed the sectional curvature of skew semi-invariant submanifolds and obtained many interesting results.

1 – Introduction

In the early years of the sixties, S. Tachibana [1] introduced and studied a class of important manifolds, i.e., locally product manifolds. After that, some authors discussed this class of manifolds, they obtained many very interesting results (cf. [2], [3], [4] and [5]). In [6], A. Bejancu defined and studied semi-invariant submanifolds of a locally product manifold. In this paper, we defined and discussed a new class of submanifolds of a locally product manifold, i.e., skew semi-invariant submanifolds, which contain semi-invariant submanifolds as a special case.

There are two parts in this paper, in section one we give the definition of skew semi-invariant submanifolds and some preliminaries which we will use later. In section two we discuss the parallelism of the canonical structures $P$ and $Q$ and the sectional curvature of skew semi-invariant submanifolds.

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2 – Definitions and preliminaries

In this paper, we suppose that all manifolds and maps are $C^\infty$-differentiable.

Let $(M, g, F)$ be an almost product Riemannian manifold, where $g$ is a Riemannian metric and $F$ is a non-trivial tensor field of type $(1, 1)$, $F$ is called an almost product structure. Moreover $g$ and $F$ satisfying the following conditions

\[(1) \quad F^2 = I \quad (F \neq \pm I), \quad g(FX, FY) = g(X, Y),\]

where $X, Y \in T\bar{M}$ and $I$ is the identity transformation.

We denote by $\nabla$ the Levi-Civita connection on $\bar{M}$ with respect to $g$, if $\nabla_X F = 0$, $X \in T\bar{M}$, we call $\bar{M}$ a locally product Riemannian manifold.

Let $M$ be a Riemannian manifold isometrically immersed in $\bar{M}$ and denote by the same symbol $g$ the Riemannian metric induced on $M$, for $p \in M$ and tangent vector $X_p \in T_p M$, we write

\[(2) \quad FX_p = PX_p + QX_p\]

where $PX_p \in T_p M$ is tangent to $M$ and $QX_p \in T^\perp_p M$ is normal to $M$.

For any two vectors $X_p, Y_p \in T_p M$, we have $g(FX_p, Y_p) = g(PX_p, Y_p)$, which implies that $g(PX_p, Y_p) = g(X_p, PY_p)$. So $P$ and $P^2$ are all symmetric operators on the tangent space $T_p M$. If $\alpha(p)$ is the eigenvalue of $P^2$ at $p \in M$, since $P^2$ is a composition of an isometry and a projection, hence $\alpha(p) \in [0, 1]$.

For each $p \in M$, we set $D^\alpha_p = \text{Ker}(P^2 - \alpha(p) I)$, where $I$ is the identity transformation on $T_p M$, and $\alpha(p)$ is an eigenvalue of $P^2$ at $p \in M$, obviously, we have $D^0_p = \text{Ker} P$, $D^1_p = \text{Ker} Q$, $D^2_p$ is the maximal $F$ invariant subspace of $T_p M$ and $D^0_p$ is the maximal $F$ anti-invariant subspace of $T_p M$. If $\alpha_1(p), ..., \alpha_k(p)$ are all eigenvalues of $P^2$ at $p$, then $T_p M$ can be decomposed as the direct sum of the mutually orthogonal eigenspaces, that is,

$$T_p M = D^{\alpha_1}_p \oplus \cdots \oplus D^{\alpha_k}_p.$$ 

Now we give the following definition.

**Definition.** A submanifold $M$ of a locally product manifold $\bar{M}$ is called a skew semi-invariant submanifold if there exists an integer $k$ and constant functions $\alpha_i$, $1 \leq i \leq k$, defined on $M$ with values in $(0, 1)$ such that

(i) Each $\alpha_i$, $1 \leq i \leq k$, is a distinct eigenvalue of $P^2$ with $T_p M = D^0_p \oplus D^1_p \oplus D^{\alpha_1}_p \oplus \cdots \oplus D^{\alpha_k}_p$, for $p \in M$.

(ii) The dimensions of $D^0_p$, $D^1_p$ and $D^{\alpha_i}_p$, $1 \leq i \leq k$, are independent of $p \in M$. 

Remark. Condition (ii) in the above definition implies that $D^0_p, D^1_p$ and $D^{α_i}_p, 1 ≤ i ≤ k$, defined $P$ invariant, mutually orthogonal distributions which we denote by $D^0, D^1$ and $D^{α_i}, 1 ≤ i ≤ k$, respectively. Moreover the tangent bundle of $M$ has the following decomposition

$$TM = D^0 \oplus D^1 \oplus D^{α_1} \oplus \cdots \oplus D^{α_k}.$$ 

Particularly if $k = 0$ then $M$ is a semi-invariant submanifold [6]. If $k = 0$, and $D^0_p(D^1_p)$ is trivial, then $M$ is an invariant (anti-invariant) submanifold of $M$ [4].

Denote the induced connection in $M$ by $\nabla$, we have the formulas of Gauss and Weingarten

$$\nabla_X Y = \nabla_X Y + h(X, Y) , \quad (3)$$

$$\nabla_X N = -A_N X + \nabla_X^N , \quad (4)$$

for all vector fields $X, Y ∈ TM$ and $N ∈ T^\perp M$. Here $h$ denotes the second fundamental form and $T^\perp M$ denotes the normal bundle of $M$ in $M$. Moreover we have

$$g(h(X, Y), N) = g(A_N X, Y) . \quad (5)$$

For $N ∈ T^\perp M$, we set

$$FN = tN + fN \quad (6)$$

where $tN ∈ TM$, $fN ∈ T^\perp M$.

From $F(\nabla_X Y) = \nabla_X FY$, (3), (4) and (6) we have

$$P(\nabla_X Y) + Q(\nabla_X Y) + t h(X, Y) + f h(X, Y) = \nabla_X P Y + h(X, PY) - A_{QY} X + \nabla_X^Q Y , \quad (7)$$

for $X, Y ∈ TM$. Comparing tangential and normal components in (7) we obtain

$$P \nabla_X Y = \nabla_X P Y - t h(X, Y) - A_{QY} X , \quad (8)$$

$$Q \nabla_X Y = h(X, PY) + \nabla_X^Q Y - f h(X, Y) , \quad (9)$$

for $X, Y ∈ TM$. From (8) and (9) we can get

$$P[X, Y] = \nabla_X P Y - \nabla_Y P X + A_{QX} Y - A_{QY} X , \quad (10)$$

$$Q[X, Y] = h(X, PY) - h(PX, Y) + \nabla_X^Q X - \nabla_X^Q X . \quad (11)$$
We have the following lemma immediately from (10) and (11)

**Lemma 1.1.** Let $M$ be a skew semi-invariant submanifold of a locally product manifold $\tilde{M}$, then

(i) The distribution $D^0$ is integrable if and only if $A_F X Y = A_F Y X$ for all $X, Y \in D^0$.

(ii) The distribution $D^1$ is integrable if and only if $h(X, FY) = h(FX, Y)$ for all $X, Y \in D^1$.

We define the covariant derivatives of $P$ and $Q$ in a manner as follows

\[(\nabla_X P) Y = \nabla_X P Y - P \nabla_X Y,\]
\[(\nabla_X Q) Y = \nabla_X^1 Q Y - Q \nabla_X Y,\]
for all $X, Y \in TM$. Using (8) and (9) we have

\[(\nabla_X P) Y = t h(X, Y) + A Q Y,\]
\[(\nabla_X Q) Y = f h(X, Y) - h(X, PY).\]

Let $D^1$ and $D^2$ be two distributions defined on a manifold $M$. We say that $D^1$ is parallel with respect to $D^2$ if for all $X \in D^2$ and $Y \in D^1$, we have $\nabla_X Y \in D^1$. $D^1$ is called parallel if for $X \in TM$ and $Y \in D^1$, we have $\nabla_X Y \in D^1$, it is easy to verify that $D^1$ is parallel if and only if the orthogonal complementary distribution of $D^1$ is also parallel.

Let $M$ be a submanifold of $\tilde{M}$. A distribution $D$ on $M$ is said to be totally geodesic if for all $X, Y \in D$ we have $h(X, Y) = 0$. In this case we say also that $M$ is $D$ totally geodesic. For two distributions $D^1$ and $D^2$ defined on $M$, we say that $M$ is $D^1$-$D^2$ mixed totally geodesic if for all $X \in D^1$ and $Y \in D^2$ we have $h(X, Y) = 0$.

**Proposition 1.1.** Let $M$ be a skew semi-invariant submanifold of a locally product manifold $\tilde{M}$, for any distribution $D^\alpha$, if $A_N PX = PA_N X$, for all $X \in D^\alpha$ and $N \in T^\perp M$, then $M$ is $D^\alpha$-$D^\beta$ mixed totally geodesic, where $\alpha \neq \beta$.

**Proof:** From the assumption we have $P^2 A_N X - \alpha A_N X = 0$, which implies that $A_N X \in D^\alpha$. So for all $Y \in D^\beta$, $N \in T^\perp M$, $\alpha \neq \beta$, we have $0 = g(A_N X, Y) = g(h(X, Y), N)$, that is $h(X, Y) = 0$, hence $M$ is $D^\alpha$-$D^\beta$ mixed totally geodesic.
From (2) and (6) we can obtain

\[ f QX_p = -Q PX_p , \]

\[ Qt N = N - f^2 N , \]

for all \( X_p \in T_p M, \ N \in T^\perp_p M \). Furthermore, for \( X_p \in D^\alpha_p, \ 1 \leq i \leq k \), we have

\[ f^2 QX_p = \alpha_i QX_p . \]

Also if \( X_p \in D^0_p \) then it is clear that \( f^2 QX_p = 0 \). Thus if \( X_p \) is an eigenvector of \( P^2 \) corresponding to the eigenvalue \( \alpha(p) \neq 1 \), then \( QX_p \) is an eigenvector of \( f^2 \) with the same eigenvalue \( \alpha(p) \). (17) implies that \( \alpha(p) \) is an eigenvalue of \( f^2 \) if and only if \( \gamma(p) = 1 - \alpha(p) \) is an eigenvalue of \( Qt \). Since \( Qt \) and \( f^2 \) are symmetric operators on the normal bundle \( T^\perp M \), their eigenspaces are orthogonal. The dimension of the eigenspace of \( Qt \) corresponding to the eigenvalue \( 1 - \alpha(p) \) is equal the dimension of \( D^\alpha_p \) if \( \alpha(p) \neq 1 \). Consequently, we have

\[ \text{Lemma 1.2.} \quad \text{Let } M \text{ be a submanifold of a locally product Riemannian manifold } \bar{M}. M \text{ is a skew semi-invariant submanifold if and only if the eigenvalues of } Qt \text{ are constant and the eigenspaces of } Qt \text{ have constant dimension.} \]

3 – Skew semi-invariant submanifold

\[ \text{Theorem 2.1.} \quad \text{Let } M \text{ be a submanifold of a locally product manifold } M, \text{ if } \nabla P = 0, \text{ then } M \text{ is a skew semi-invariant submanifold. Furthermore each of the } P \text{ invariant distributions } D^0, D^1 \text{ and } D^\alpha, \ 1 \leq i \leq k, \text{ is parallel.} \]

\[ \text{Proof:} \quad \text{Fix } p \in M, \text{ for any } Y_p \in D^\alpha_p \text{ and any vector field } X \in TM, \text{ let } Y \text{ be the parallel translation of } Y_p \text{ along the integral curve of } X. \text{ Since } (\nabla_X P) Y = 0, \text{ we have by (8)} \]

\[ \nabla_X (P^2 - \alpha(p) Y) = P^2 \nabla_X Y - \alpha(p) \nabla_X Y = 0 \]

since \( P^2 Y - \alpha(p) Y = 0 \) at \( p \), it is identical 0 on \( M \). Thus the eigenvalues of \( P^2 \) are constant. Moreover, parallel translation of \( T_p M \) along any curve is an isometry which preserves each \( D^\alpha \). Thus the dimension of each \( D^\alpha \) is constant and \( M \) is a skew semi-invariant submanifold.

Now if \( Y \) is any vector field in \( D^\alpha \), we have \( P^2 Y = \alpha Y \ (\alpha \text{ constant}), \) i.e., \( P^2 \nabla_X Y = \alpha \nabla_X Y \) which implies that \( D^\alpha \) is parallel. ■
Next we turn our attention to the vanishing of $\nabla Q$. For $X, Y \in TM$, if $(\nabla_X Q)Y = 0$ then (15) yields
\begin{equation}
fh(X, Y) = h(X, PY).
\end{equation}
In particular, if $Y \in D^\alpha$ then (19) implies
\begin{equation}
f^2 h(X, Y) = \alpha h(X, Y)
\end{equation}
consequently we have

**Proposition 2.1.** Let $M$ be a skew semi-invariant submanifold of a locally product manifold $\tilde{M}$, if $\nabla Q \equiv 0$, then $M$ is $D^\alpha-D^\beta$ mixed totally geodesic for all $\alpha \neq \beta$. Moreover, if $X \in D^\alpha$ then either $h(X, X) = 0$ or $h(X, X)$ is an eigenvector of $f^2$ with eigenvalue $\alpha$.

The next lemma is easy to prove so we omit the proof.

**Lemma 2.1.** Let $M$ be a submanifold of a locally product manifold $\tilde{M}$, then $\nabla Q = 0$ if and only if $\nabla_X tN = t \nabla_X ^1 N$ for all $X \in TM$ and $N \in T^\perp M$.

**Theorem 2.2.** Let $M$ be a submanifold of a locally product manifold $\tilde{M}$, if $\nabla Q = 0$, then $M$ is skew semi-invariant submanifold.

**Proof:** If $TM = D^1$ then we are done. Otherwise, we may find a point $p \in M$ and a vector $X_p \in D^\alpha_p$, $\alpha \neq 1$. Set $N_p = QX_p$, then $N_p$ is an eigenvector of $Qt$ with eigenvalue $\gamma(p) = 1 - \alpha(p)$. Now, let $Y \in TM$ and $N$ be the translation of $N_p$ in the normal bundle $T^\perp M$ along an integral curve of $Y$, we have
\begin{align*}
\nabla_Y ^1 (Qt N - \gamma(p) N) &= \nabla_Y ^1 Qt N - \gamma(p) \nabla_Y ^1 N = Q(\nabla_Y tN) - \gamma(p) \nabla_Y ^1 N.
\end{align*}
By Lemma 2.1, this becomes $\nabla_Y ^1 (Qt N - \gamma(p) N) = Qt \nabla_Y ^1 N - \gamma(p) \nabla_Y ^1 N = 0$. Since $Qt N - \gamma(p) N = 0$ at $p$, $Qt N - \gamma(p) N \equiv 0$ on $M$. It follows from Lemma 1.2 that $M$ is a skew semi-invariant submanifold. $lacksquare$

For a submanifold $M$ of a locally product manifold $\tilde{M}$, let $\tilde{R}$ (resp. $R$) denote the curvature tensor of $\tilde{M}$ (resp. $M$), then the equation of Gauss is given by
\begin{align}
g(R(X, Y) Z, W) &= g(\tilde{R}(X, Y) Z, W) + g(h(X, W), h(Y, Z)) \\
&\quad - g(h(X, Z), h(Y, W))
\end{align}
for $X, Y, Z, W \in TM$. 

The sectional curvature of a plane section of $\tilde{M}$ determined by two orthogonal unit vectors $X, Y \in T\tilde{M}$ is given by

$$(22) \quad K_{\tilde{M}}(X \wedge Y) = g\left(\tilde{R}(X, Y) Y, X\right).$$

The sectional curvature of a plane section of $M$ determined by two orthogonal unit vectors $X, Y \in TM$ is given by

$$(23) \quad K_M(X \wedge Y) = g\left(R(X, Y) Y, X\right).$$

For $X, Y \in TM$, from (21), (22) and (23) we can obtain

$$(24) \quad K_{\tilde{M}}(X \wedge Y) = K_M(X \wedge Y) + g\left(h(X, X), h(Y, Y)\right) - |h(X, Y)|^2.$$

**Proposition 2.2.** Let $M$ be a skew semi-invariant submanifold of a locally product manifold $\tilde{M}$, if $\nabla Q = 0$, then for any unit vectors $X \in D^\alpha$ and $Y \in D^\beta$, $\alpha \neq \beta$, we have $K_M(X \wedge Y) = K_{\tilde{M}}(X \wedge Y)$.

**Proof:** It can be followed easily from Proposition 2.1.

**Lemma 2.2.** Let $M$ be a skew semi-invariant submanifold of a locally product manifold $\tilde{M}$, then the followings are equivalent

(i) $(\nabla_X Q) Y - (\nabla_Y Q) X = 0$ for all $X, Y \in D^\alpha$.

(ii) $h(P, X, Y) = h(X, PY)$ for all $X, Y \in D^\alpha$.

(iii) $Q[X, Y] = \nabla_X QY - \nabla_Y QX$ for all $X, Y \in D^\alpha$.

(iv) $A_N PY - PA_N Y$ is perpendicular to $D^\alpha$ for all $Y \in D^\alpha$ and $N \in T^\perp N$.

The proof is very trivial, we omit it here.

We call $P \alpha$ commutative if any of the equivalent conditions in the above Lemma holds.

For each $P$ invariant $D^\alpha$, let $n(\alpha) = \dim D^\alpha$. For each $D^\alpha$ we may choose a local orthonormal basis $E^1, \ldots, E^{n(\alpha)}$. Define the $D^\alpha$ mean curvature vector by $H^\alpha = \sum_{i=1}^{n(\alpha)} h(E^i, E^i)$, then the mean curvature vector is given by $H = \frac{1}{n} (H^0 + H^1 + H^{\alpha_1} + \cdots + H^{\alpha_k})$, $n = \dim M$.

A skew semi-invariant submanifold $M$ of a locally product manifold $\tilde{M}$ is called $D^\alpha$ minimal if $H^\alpha = 0$ and minimal if $H = 0$.

For any unit vector $X \in D^\alpha$, $\alpha \neq 0$, defined the $\alpha$ sectional curvature of $\tilde{M}$ and $M$ by

$\bar{H}_\alpha(X) = K_{\tilde{M}}(X \wedge Y), \quad H_\alpha(X) = K_M(X \wedge Y)$
respectively, where $Y = \frac{PX}{\sqrt{\alpha}}$. From (24) we have

\begin{equation}
H_\alpha(X) = \bar{H}_\alpha(X) - \frac{1}{\alpha} g\left( h(X, X), h(PX, PX) \right) - \frac{1}{\alpha} |h(X, PX)|^2.
\end{equation}

Then we have the following proposition

**Proposition 2.3.** Let $M$ be a skew semi-invariant submanifold of a locally product manifold $\tilde{M}$, if $P$ is $\alpha$ commutative, $\alpha \neq 0$, then

\begin{equation}
H_\alpha(X) = \bar{H}_\alpha(X) + |h(X, X)|^2 - \frac{1}{\alpha} |h(X, PX)|^2.
\end{equation}

Let $\{E^1, ..., E^{n(\alpha)}\}$ and $\{F^1, ..., F^{n(\beta)}\}$ be the local orthonormal bases for $D^\alpha$ and $D^\beta$, respectively. We define $\alpha$-$\beta$ sectional curvatures of $\tilde{M}$ and $M$ by

\[
\bar{\rho}_{\alpha\beta} = \sum_{i=1}^{n(\alpha)} \sum_{j=1}^{n(\beta)} K_{\tilde{M}}(E^i \wedge F^j), \quad \rho_{\alpha\beta} = \sum_{i=1}^{n(\alpha)} \sum_{j=1}^{n(\beta)} K_M(E^i \wedge F^j),
\]

respectively.

From (24) we see that for $\alpha \neq \beta$ we have

\begin{equation}
\rho_{\alpha\beta} = \bar{\rho}_{\alpha\beta} + g(H^\alpha, H^\beta) - \sum_{i=1}^{n(\alpha)} \sum_{j=1}^{n(\beta)} |h(E^i \wedge F^j)|^2,
\end{equation}

for $\alpha = \beta$ we have

\begin{equation}
\rho_{\alpha\alpha} = \bar{\rho}_{\alpha\alpha} - \sum_{i=1}^{n(\alpha)} \sum_{j=1}^{n(\beta)} |h(E^i \wedge F^j)|^2.
\end{equation}

Using (26) and (27) we have the following proposition

**Proposition 2.4.** Let $M$ be a skew semi-invariant submanifold of a locally product manifold $\tilde{M}$.

(i) If $H^\alpha$ is perpendicular to $H^\beta$, $\alpha \neq \beta$, then $\rho_{\alpha\beta} \leq \bar{\rho}_{\alpha\beta}$, and the equality holds if and only if $M$ is $D^\alpha$-$D^\beta$ mixed totally geodesic.

(ii) If $M$ is $D^\alpha$ minimal, then $\rho_{\alpha\alpha} \leq \bar{\rho}_{\alpha\alpha}$, and the equality holds if and only if $M$ is $D^\alpha$ totally geodesic.
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