

## MORE ON SG-COMPACT SPACES \*

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**Abstract:** The aim of this paper is to continue the study of sg-compact spaces, a topological notion much stronger than hereditary compactness. We investigate the relations between sg-compact and  $C_2$ -spaces and the interrelations to hereditarily sg-closed sets.

### 1 – Introduction

In 1995, sg-compact spaces were introduced independently by Caldas [2] and by Devi, Balachandran and Maki [4]. A topological space  $(X, \tau)$  is called *sg-compact* [2] if every cover of  $X$  by sg-open sets has a finite subcover. In [4], the term *SGO-compact* is used.

Recall that a subset  $A$  of a topological space  $(X, \tau)$  is called *sg-open* [1] if every semi-closed subset of  $A$  is included in the semi-interior of  $A$ . A set  $A$  is called *semi-open* if  $A \subseteq \overline{\text{Int } A}$  and *semi-closed* if  $\text{Int } \overline{A} \subseteq A$ . The *semi-interior* of  $A$ , denoted by  $\text{sInt}(A)$ , is the union of all semi-open subsets of  $A$  while the *semi-closure* of  $A$ , denoted by  $\text{sCl}(A)$ , is the intersection of all semi-closed supersets of  $A$ . It is well known that  $\text{sInt}(A) = A \cap \overline{\text{Int } A}$  and  $\text{sCl}(A) = A \cup \text{Int } \overline{A}$ .

Every topological space  $(X, \tau)$  has a unique decomposition into two sets  $X_1$  and  $X_2$ , where  $X_1 = \{x \in X : \{x\} \text{ is nowhere dense}\}$  and  $X_2 = \{x \in X : \{x\} \text{ is locally dense}\}$ . This decomposition follows from a result of Janković and Reilly [13, Lemma 2]. Recall that a set  $A$  is said to be *locally dense* [3] (= *preopen*) if  $A \subseteq \text{Int } \overline{A}$ .

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It is a fact that a subset  $A$  of  $X$  is sg-closed (= its complement is sg-open) if and only if  $X_1 \cap \text{sCl}(A) \subseteq A$  [6], or equivalently if and only if  $X_1 \cap \text{Int } \overline{A} \subseteq A$ . By taking complements one easily observes that  $A$  is sg-open if and only if  $A \cap X_1 \subseteq \text{sInt}(A)$ . Hence every subset of  $X_2$  is sg-open.

## 2 – Sg-compact spaces

Let  $A$  be a sg-closed subset of a topological space  $(X, \tau)$ . If every subset of  $A$  is also sg-closed in  $(X, \tau)$ , then  $A$  will be called *hereditarily sg-closed* (= hsg-closed). Observe that every nowhere dense subset is hsg-closed but not vice versa.

**Proposition 2.1.** *For a subset  $A$  of a topological space  $(X, \tau)$  the following conditions are equivalent:*

- (1)  $A$  is hsg-closed.
- (2)  $X_1 \cap \text{Int } \overline{A} = \emptyset$ .

**Proof.** (1) $\Rightarrow$ (2) Suppose that there exists  $x \in X_1 \cap \text{Int } \overline{A}$ . Let  $V_x$  be an open set such that  $V_x \subseteq \overline{A}$  and let  $B = A \setminus \{x\}$ . Since  $B$  is sg-closed, i.e.  $X_1 \cap \text{sCl}(B) \subseteq B$ , we have  $x \notin \text{sCl}(B)$ , hence  $x \notin \text{Int } \overline{B}$ , and thus  $x \in X \setminus \overline{B}$ . If  $H = V_x \cap (X \setminus \overline{B})$ , then  $H$  is nonempty and open with  $H \subseteq \overline{A}$  and  $H \cap B = \emptyset$  and so  $H \cap A = \{x\}$ . Hence  $\emptyset \neq H = H \cap \overline{A} \subseteq \overline{H \cap A} \subseteq \overline{\{x\}}$ , i.e.  $\text{Int}\{x\} \neq \emptyset$ . Thus  $x \in X_2$ , a contradiction.

(2) $\Rightarrow$ (1) Let  $B \subseteq A$ . Then  $\text{Int } \overline{B} \subseteq \text{Int } \overline{A}$  and  $X_1 \cap \text{Int } \overline{B} = \emptyset$ , i.e.  $B$  is sg-closed. ■

We will call a topological space  $(X, \tau)$  a  $C_2$ -space [9] (resp.  $C_3$ -space) if every nowhere dense (resp. hsg-closed) set is finite. Clearly every  $C_3$ -space is a  $C_2$ -space. Also, a topological space  $(X, \tau)$  is indiscrete if and only if every subset of  $X$  is hsg-closed (since in that case  $X_1 = \emptyset$ ).

Following Hodel [14], we say that a *cellular family* in a topological space  $(X, \tau)$  is a collection of nonempty, pairwise disjoint open sets. The following result reveals an interesting property of  $C_2$ -spaces.

**Lemma 2.2.** *Let  $(X, \tau)$  be a  $C_2$ -space. Then every infinite cellular family has an infinite subfamily whose union is contained in  $X_2$ .*

**Proof.** Let  $\{U_i : i \in \mathbb{N}\}$  be a cellular family. Suppose that for infinitely many  $i \in \mathbb{N}$  we have  $U_i \cap X_1 \neq \emptyset$ . Without loss of generality we may assume that  $U_i \cap X_1 \neq \emptyset$  for each  $i \in \mathbb{N}$ . Now pick  $x_i \in U_i \cap X_1$  for each  $i \in \mathbb{N}$  and partition  $\mathbb{N}$  into infinitely many disjoint infinite sets,  $\mathbb{N} = \bigcup_{k \in \mathbb{N}} \mathbb{N}_k$ . Let  $A_k = \{x_i : i \in \mathbb{N}_k\}$ . Since  $A_k \cap (\bigcup_{i \notin \mathbb{N}_k} U_i) = \emptyset$  and  $A_k \subseteq \bigcup_{i \in \mathbb{N}_k} U_i$  for each  $k$ , it is easily checked that  $\{\text{Int } \overline{A_k} : k \in \mathbb{N}\}$  is a disjoint family of open sets. Since  $X$  is a  $C_2$ -space,  $A_k$  cannot be nowhere dense and so, for each  $k$ , there exists  $p_k \in \text{Int } \overline{A_k}$  and the  $p_k$ 's are pairwise distinct. Also, since  $X$  is  $C_2$ ,  $\overline{\bigcup_{i \in \mathbb{N}} U_i} = \bigcup_{i \in \mathbb{N}} (U_i) \cup F$ , where  $F$  is finite. Since  $p_k \in \overline{\bigcup_{i \in \mathbb{N}} U_i}$  for each  $k$ , there exists  $k_0$  such that  $p_k \in \bigcup_{i \in \mathbb{N}} U_i$  for  $k \geq k_0$ , and since  $\text{Int } \overline{A_k} \cap (\bigcup_{i \notin \mathbb{N}_k} U_i) = \emptyset$ , we have  $p_k \in \bigcup_{i \in \mathbb{N}_k} U_i$  for  $k \geq k_0$ . Now, for each  $k \geq k_0$  pick  $i_k \in \mathbb{N}_k$  such that  $p_k \in U_{i_k}$ , and so  $p_k \in W = U_{i_k} \cap \text{Int } \overline{A_k}$ . Thus  $\emptyset \neq W \subseteq U_{i_k} \cap \overline{A_k} \subseteq \overline{U_{i_k} \cap A_k} = \overline{\{x_{i_k}\}}$ . Hence  $\{x_{i_k}\}$  is locally dense, a contradiction. This shows that only for finitely many  $i \in \mathbb{N}$  we have  $U_i \cap X_1 \neq \emptyset$ . Thus the claim is proved. ■

The  $\alpha$ -topology [16] on a topological space  $(X, \tau)$  is the collection of all sets of the form  $U \setminus N$ , where  $U \in \tau$  and  $N$  is nowhere dense in  $(X, \tau)$ . Recall that topological spaces whose  $\alpha$ -topologies are hereditarily compact have been shown to be *semi-compact* [11]. The original definition of semi-compactness is in terms of semi-open sets and is due to Dorsett [8]. By definition a topological space  $(X, \tau)$  is called *semi-compact* [8] if every cover of  $X$  by semi-open sets has a finite subcover.

**Remark 2.3.**

- (i) The 1-point-compactification of an infinite discrete space is a  $C_2$ -space having an infinite cellular family.
- (ii) [9] A topological space  $(X, \tau)$  is semi-compact if and only if  $X$  is a  $C_2$ -space and every cellular family is finite.
- (iii) [12] Every subspace of a semi-compact space is semi-compact (as a subspace).

**Lemma 2.4.**

- (i) Every  $C_3$ -space  $(X, \tau)$  is semi-compact.
- (ii) Every sg-compact space is semi-compact.

**Proof.** (i) All  $C_3$ -spaces are  $C_2$ -spaces. Thus in the notion of Remark 2.3 (ii) above we need to show that every cellular family in  $X$  is finite. Suppose that there exists an infinite cellular family  $\{U_i : i \in \mathbb{N}\}$ . For each  $i \in \mathbb{N}$  pick  $x_i \in U_i$

and, as before, partition  $\mathbb{N} = \bigcup_k \mathbb{N}_k$  and set  $A_k = \{x_i : i \in \mathbb{N}_k\}$ . Since  $X$  is a  $C_2$ -space,  $\{\text{Int } \overline{A_k} : k \in \mathbb{N}\}$  is a cellular family. By Lemma 2.2, there is a  $k \in \mathbb{N}$  such that  $\text{Int } \overline{A_k} \subseteq X_2$ . Since  $A_k$  is not hsg-closed, we must have  $X_1 \cap \text{Int } \overline{A_k} \neq \emptyset$ , a contradiction. So, every cellular family in  $X$  is finite and consequently  $(X, \tau)$  is semi-compact.

(ii) is obvious since every semi-open set is sg-open. ■

**Remark 2.5.**

- (i) It is known that sg-open sets are  $\beta$ -open, i.e. they are dense in some regular closed subspace [5]. Note that  $\beta$ -compact spaces, i.e. the spaces in which every cover by  $\beta$ -open sets has a finite subcover are finite [10]. However, one can easily find an example of an infinite sg-compact space — the real line with the cofinite topology is such a space.
- (ii) In semi- $T_D$ -spaces the concepts of sg-compactness and semi-compactness coincide. Recall that a topological space  $(X, \tau)$  is called a *semi- $T_D$ -space* [13] if each singleton is either open or nowhere dense, i.e. if every sg-closed set is semi-closed.

**Theorem 2.6.** *For a topological space  $(X, \tau)$  the following conditions are equivalent:*

- (1)  $X$  is sg-compact.
- (2)  $X$  is a  $C_3$ -space.

**Proof.** (1) $\Rightarrow$ (2) Suppose that there exists an infinite hsg-closed set  $A$  and set  $B = X \setminus A$ . Observe that for each  $x \in A$ , the set  $B \cup \{x\}$  is sg-open in  $X$ . Thus  $\{B \cup \{x\} : x \in A\}$  is a sg-open cover of  $X$  with no finite subcover. Thus  $(X, \tau)$  is  $C_3$ .

(2) $\Rightarrow$ (1) Let  $X = \bigcup_{i \in I} A_i$ , where each  $A_i$  is sg-open. Let  $S_i = \text{sInt}(A_i)$  for each  $i \in I$  and let  $S = \bigcup_{i \in I} S_i$ . Then  $S$  is a semi-open subset of  $X$  and each  $S_i$  is a semi-open subset of  $(S, \tau|_S)$ . Since  $X$  is a  $C_3$ -space,  $(X, \tau)$  is semi-compact and hence  $(S, \tau|_S)$  is a semi-compact subspace of  $X$  (by Remark 2.3 (iii)). So we may say that  $S = S_{i_1} \cup \dots \cup S_{i_k}$ . Since  $A_i$  is sg-open, we have  $X_1 \cap A_i \subseteq S_i$  for each index  $i$  and so  $X_1 = X_1 \cap (\bigcup A_i) \subseteq X_1 \cap S \subseteq S_{i_1} \cup \dots \cup S_{i_k} = S$ . Hence  $X \setminus S$  is semi-closed and  $X \setminus S \subseteq X_2$ . Since  $\text{Int}(\overline{X \setminus S}) \subseteq X \setminus S \subseteq X_2$ , we conclude that  $X \setminus S$  is hsg-closed and thus finite. This shows that  $X = S_{i_1} \cup \dots \cup S_{i_k} \cup (X \setminus S) = A_{i_1} \cup \dots \cup A_{i_k} \cup F$ , where  $F$  is finite, i.e.  $(X, \tau)$  is sg-compact. ■

**Remark 2.7.**

- (i) If  $X_1 = X$ , then  $(X, \tau)$  is sg-compact if and only if  $(X, \tau)$  is semi-compact. Observe that in this case sg-closedness and semi-closedness coincide.
- (ii) Every infinite set endowed with the cofinite topology is (hereditarily) sg-compact.

It is known that an arbitrary intersection of sg-closed sets is also an sg-closed set [6]. The following result provides an answer to the question about the additivity of sg-closed sets.

**Proposition 2.8.**

- (i) If  $A$  is sg-closed and  $B$  is closed, then  $A \cup B$  is also sg-closed.
- (ii) The intersection of a sg-open and an open set is always sg-open.
- (iii) The union of a sg-closed and a semi-closed set need not be sg-closed, in particular, even finite union of sg-closed sets need not be sg-closed.

**Proof.** (i) Let  $A \cup B \subseteq U$ , where  $U$  is semi-open. Since  $A$  is sg-closed, we have  $sCl(A \cup B) = (A \cup B) \cup Int(\overline{A \cup B}) \subseteq U \cup Int(\overline{A \cup B}) \subseteq U \cup (Int \overline{A} \cup B) \subseteq U \cup (U \cup B) = U$ .

(ii) follows from (i).

(iii) Let  $X = \{a, b, c, d\}$ ,  $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ . Note that the two sets  $A = \{a\}$  and  $B = \{b\}$  are semi-closed but their union  $\{a, b\}$  is not sg-closed. ■

Theorem 3 from [1] states that if  $B \subseteq A \subseteq (X, \tau)$  and  $A$  is open and sg-closed, then  $B$  is sg-closed in the subspace  $A$  if and only if  $B$  is sg-closed in  $X$ . Since a subset is regular open if and only if it is  $\alpha$ -open and sg-closed [7], by using Proposition 2.8, we obtain the following result:

**Proposition 2.9.** *Let  $R$  be a regular open subset of a topological space  $(X, \tau)$ . If  $A \subseteq R$  and  $A$  is sg-open in  $(R, \tau|_R)$ , then  $A$  is sg-open in  $X$ .*

**Proof.** Since  $B = R \setminus A$  is sg-closed in  $(R, \tau|_R)$ ,  $B$  is sg-closed in  $X$  by [1, Theorem 3]. Thus  $X \setminus B$  is sg-open in  $X$  and by Proposition 2.8 (ii),  $R \cap (X \setminus B) = A$  is sg-open in  $X$ . ■

Recall that a subset  $A$  of a topological space  $(X, \tau)$  is called  $\delta$ -open [18] if  $A$  is a union of regular open sets. The collection of all  $\delta$ -open subsets of a topological space  $(X, \tau)$  forms the so called *semi-regularization topology*.

**Corollary 2.10.** *If  $A \subseteq B \subseteq (X, \tau)$  such that  $B$  is  $\delta$ -open in  $X$  and  $A$  is sg-open in  $B$ , then  $A$  is sg-open in  $X$ .*

**Proof.** Let  $B = \bigcup_{i \in I} B_i$ , where each  $B_i$  is regular open in  $(X, \tau)$ . Clearly, each  $B_i$  is regular open also in  $(B, \tau|_B)$ . By Proposition 2.8 (ii),  $A \cap B_i$  is sg-open in  $(B, \tau|_B)$  for each  $i \in I$ . In the notion of Proposition 2.9,  $B \setminus (A \cap B_i)$  is sg-closed in  $(X, \tau)$  for each  $i \in I$ . Hence  $X \setminus (B \setminus (A \cap B_i)) = (A \cap B_i) \cup (X \setminus B)$  is sg-open in  $(X, \tau)$ . Again by Proposition 2.8 (ii),  $B \cap ((A \cap B_i) \cup (X \setminus B)) = A \cap B_i$  is sg-open in  $(X, \tau)$ . Since any union of sg-open sets is always sg-open, we have  $A = \bigcup_{i \in I} (A \cap B_i)$  is sg-open in  $(X, \tau)$ . ■

**Proposition 2.11.** *Every  $\delta$ -open subset of a sg-compact space  $(X, \tau)$  is sg-compact, in particular, sg-compactness is hereditary with respect to regular open sets.*

**Proof.** Let  $A \subseteq X$  be  $\delta$ -open. If  $\{U_i : i \in I\}$  is a sg-open cover of  $(S, \tau|_S)$ , then by Corollary 2.10, each  $U_i$  is sg-open in  $X$ . Then,  $\{U_i : i \in I\}$  along with  $X \setminus A$  forms a sg-open cover of  $X$ . Since  $X$  is sg-compact, there exists a finite  $F \subseteq I$  such that  $\{U_i : i \in F\}$  covers  $A$ . ■

**Example 2.12.** Let  $A$  be an infinite set with  $p \notin A$ . Let  $X = A \cup \{p\}$  and  $\tau = \{\emptyset, A, X\}$ .

- (i) Clearly,  $X_1 = \{p\}$ ,  $X_2 = A$  and for each infinite  $B \subseteq X$ , we have  $\overline{B} = X$ . Hence  $X_1 \cap \text{Int} \overline{B} \neq \emptyset$ , so  $B$  is not hsg-closed. Thus  $(X, \tau)$  is a  $C_3$ -space, so sg-compact. But the open subspace  $A$  is an infinite indiscrete space which is not sg-compact. This shows that (1) hereditary sg-compactness is a strictly stronger concept than sg-compactness and (2) in Proposition 2.11 ‘ $\delta$ -open’ cannot be replaced with ‘open’.
- (ii) Observe that  $X \times X$  contains an infinite nowhere dense subset, namely  $X \times X \setminus A \times A$ . This shows that even the finite product of two sg-compact spaces need not be sg-compact, not even a  $C_2$ -space.
- (iii) [15] If the nonempty product of two spaces is sg-compact  $T_{gs}$ -space (see [15]), then each factor space is sg-compact.

Recall that a function  $f : (X, \tau) \rightarrow (Y, \sigma)$  is called *pre-sg-continuous* [17] if  $f^{-1}(F)$  is sg-closed in  $X$  for every semi-closed subset  $F \subseteq Y$ .

**Proposition 2.13.**

- (i) The property 'sg-compact' is topological.
- (ii) Pre-sg-continuous images of sg-compact spaces are semi-compact. ■

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