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# AN APPROXIMATION PROCEDURE FOR FIXED POINTS OF STRONGLY LIPSCHITZ OPERATORS

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**Abstract:** Based on a modified iterative algorithm, fixed points of the operators of the form $S = T + U$ on nonempty closed convex subsets of Hilbert spaces are approximated. Here $T$ is strongly Lipschitz and Lipschitz continuous and $U$ is Lipschitz continuous.

## 1 – Introduction

Recently, Wittmann [5, Theorem 2] approximated fixed points of nonexpansive mappings $T$ on nonempty closed convex subsets of Hilbert spaces by employing an iterative procedure

$$x_n = (1-a_n)\, x_0 + a_n\, T x_{n-1} \quad \text{for} \ \ n \geq 1 \ , \tag{1}$$

where $\{a_n\}$ is an increasing sequence in $[0,1)$ such that

$$\lim_{n\to\infty} a_n = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} (1-a_n) = \infty \ . \tag{2}$$

This result, for example, applies to $a_n = 1 - n^{-a}$ with $0 < a \leq 1$, and improves a theorem of Halpern [1, Theorem 3] which does not apply to the case $a_n = 1 - 1/n$. Furthermore, (2) is not just sufficient, but also necessary for the convergence of $\{x_n\}$ for all $T$ [1, Theorem 2].

Here we are concerned with the approximation of fixed points of operators of the form $S = T + U$, where $T$ is strongly Lipschitz and Lipschitz continuous and $U$ is Lipschitz continuous on a nonempty closed convex subset $K$ of a real Hilbert

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space $H$ by using the following modified iterative procedure in a more general setting

$$x_{n+1} = (1-a_n)\,x_n + a_n\Big[(1-t)\,x_n + t(T+U)\,x_n\Big] \quad \text{for } \; n \geq 0 \;, \tag{3}$$

where $t > 0$ is arbitrary and the sequence $\{x_n\}$ lies in $[0,1]$ such that $\sum_{n=0}^{\infty} a_n$ diverges for all $n \geq 0$.

For $U = 0$ in (3), we find the iterative algorithm

$$x_{n+1} = (1-a_n)\,x_n + a_n\Big[(1-t)\,x_n + t\,Tx_n\Big] \quad \text{for all } \; n \geq 0 \;. \tag{4}$$

For $t = 1$ in (4), we arrive at

$$x_{n+1} = (1-a_n)\,x_n + a_n\,Tx_n \quad \text{for all } \; n \geq 0 \;. \tag{5}$$

## 2 – Preliminaries

Let $H$ be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$.

**Definition 2.1** An operator $T\colon\, H \to H$ is said to be strongly Lipschitz if, for all $u, v$ in $H$, there exists a real number $r \geq 0$ such that

$$\langle Tu - Tv,\, u - v\rangle \leq -r\|u - v\|^2 \;. \tag{6}$$

The operator $T$ is called Lipschitz continuous if there exists a real number $s > 0$ such that

$$\|Tu - Tv\| \leq s\|u - v\| \quad \text{for all } \; u, v \; \text{ in } H \;. \tag{7}$$

It is easily seen that (7) implies that

$$\langle Tu - Tv,\, u - v\rangle \leq s\|u - v\|^2 \quad \text{for all } \; u, v \; \text{ in } H \;. \tag{8}$$

**Definition 2.2.** An operator $T\colon\, H \to H$ is said to be hemicontinuous if, for all $u, v$ in $H$, the function

$$t \to \Big\langle T(tu + (1-t)\,v),\, u - v\Big\rangle \quad \text{for } \; 0 \leq t \leq 1 \;, \tag{9}$$

is continuous.

To this end, let us consider an example of strongly Lipschitz operator where the real number $r$ in inequality (6) is slightly relaxed.

**Example 2.1** [6]**:** Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T\colon K \to K$ be hemicontinuous on $K$ such that, for a real number $r > -1$ and for all $u, v$ in $K$,

$$\langle Tu - Tv,\, u - v\rangle \le -r\|u-v\|^2 \ . \tag{10}$$

Then $T$ has a unique fixed point in $K$.

## 3 – The fixed point theorem

In this section we consider the approximation of fixed points of a combination of strongly Lipschitz and Lipschitz continuous operators.

**Theorem 3.1.** *Let $H$ be a real Hilbert space and let $K$ be a non-empty closed convex subset of $H$. Let $T\colon K \to K$ be strongly Lipschitz and Lipschitz continuous with respective real numbers $r \ge 0$ and $s \ge 1$, and let $U\colon K \to K$ be Lipschitz continuous with a real number $m > 0$. Let $F$ be a nonempty set of fixed points of $S = T + U$, and let $\{a_n\}$ be a sequence in $[0,1]$ such that $\sum_{n=0}^{\infty} a_n$ diverges for all $n \ge 0$. Then, for any $x_0$ in $K$, the sequence $\{x_n\}$ generated by the iterative algorithm (3) for*

$$0 \le k = \left[\left((1-t)^2 - 2t(1-t)\,r + t^2\,s^2\right)^{1/2} + t\,m\right] < 1 \tag{11}$$

*for all $t$ such that $0 < t < 2(1+r-m)/(1+2r+s^2-m^2)$ and $1+r-m > 0$, converges to a fixed point of $S = T + U$.*

When $U = 0$ in Theorem 3.1 , we arrive at the following result.

**Corollary 3.1.** *Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T\colon K \to K$ be strongly Lipschitz and Lipschitz continuous with corresponding constants $r \ge 0$ and $s \ge 1$. Let $\{a_n\}$ be a sequence in $[0,1]$ such that $\sum_{n=0}^{\infty} a_n$ diverges for all $n \ge 0$. Then, for any element $x_0$ in $K$, the sequence $\{x_n\}$ generated by the iterative algorithm (4) for*

$$0 \le k_0 = \left[(1-t)^2 - 2t(1-t)\,r + t^2\,s^2\right]^{1/2} < 1 \tag{12}$$

*for all $t$ such that $0 < t < 2(1+r)/(1+2r+s^2)$, converges to a fixed point of $T$.*

**Proof of Theorem 3.1:** For an element $z$ in $F$, we have

$$\|x_{n+1}-z\| = \tag{13}$$

$$= \Big\|(1-a_n)\,x_n + a_n\Big[(1-t)\,x_n + t(T+U)\,x_n\Big] - (1-a_n)\,z$$

$$- a_n\Big[(1-t)\,z + t(T+U)\,z\Big]\Big\|$$

$$= \Big\|(1-a_n)\,(x_n-z) + a_n\Big[(1-t)\,(x_n-z) + t(Tx_n-Tz) + t(Ux_n-Uz)\Big]\Big\|$$

$$\le (1-a_n)\,\|x_n-z\| + a_n\Big\|(1-t)\,(x_n-z) + t(Tx_n-Tz)\Big\| + a_n\,t\|Ux_n-Uz\| \;.$$

Since $T$ is strongly Lipschitz and Lipschitz continuous, this implies that

$$\Big\|(1-t)\,(x_n-z) + t(Tx_n-Tz)\Big\|^2 = \tag{14}$$

$$= (1-t)^2\,\|x_n-z\|^2 + 2t(1-t)\,\langle Tx_n-Tz,\, x_n-z\rangle + t^2\|Tx_n-Tz\|^2$$

$$\le \Big[(1-t)^2 - 2t(1-t)\,r + t^2\,s^2\Big]\,\|x_n-z\|^2 \;.$$

Applying (14) to (13) and using the Lipschitz continuity of $U$, it follows that

$$\|x_{n+1}-z\| \le \bigg\{(1-a_n) + \Big[\Big((1-t)^2 - 2t(1-t)\,r + t^2\,s^2\Big)^{1/2} + t\,m\Big]\,a_n\bigg\}\,\|x_n-z\|$$

$$= \Big[1-(1-k)\,a_n\Big]\,\|x_n-z\|$$

$$\le \prod_{j=0}^{n}\Big[1-(1-k)\,a_j\Big]\,\|x_0-z\| \;,$$

where $0 \le k = [((1-t)^2 - 2t(1-t)r + t^2s^2)^{1/2} + tm] < 1$ for all $t$ such that $0 < t < 2(1+r-m)/(1+2r+s^2-m^2)$ for $1+r-m > 0$. Since $\sum_{j=0}^{\infty} a_j$ diverges and $k<1$, this implies that $\lim_{n\to\infty}\prod_{j=0}^{n}[1-(1-k)\,a_j] = 0$, and consequently, $\{x_n\}$ converges to $z$, a fixed point of $S = T+U$. This completes the proof. ∎

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