A REMARK ON PARABOLIC EQUATIONS

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Abstract: If $L = L^*$ is a self-adjoint linear operator generating a strongly continuous semi-group on a real Hilbert space $H$ and $\alpha \in L^\infty(\mathbb{R}^+)$, any mild solution $u$ of $u' = Lu + \alpha(t)u$ satisfies $(u(0), u(t)) \geq 0$ for all $t \geq 0$. On the other hand for any $\lambda > 0$ such that $(\pi/L)^2 < \lambda < 4(\pi/L)^2$, there are solutions $u$ of the one-dimensional semilinear heat equation $u_t - u_{xx} + u^3 - \lambda u = 0$ in $\mathbb{R}^+ \times (0, L)$, $u(t, 0) = u(t, L) = 0$ on $\mathbb{R}^+$ such that $\int_\Omega u(0, x) u(t, x) \, dx < 0$ for some $t > 0$.

Résumé: Si $L = L^*$ est un opérateur auto-adjoint, générateur d’un semi-groupe fortement continu sur un espace de Hilbert réel $H$ et $\alpha \in L^\infty(\mathbb{R}^+)$, toute solution $u$ de $u' = Lu + \alpha(t)u$ satisfait $(u(0), u(t)) \geq 0$ pour tout $t \geq 0$. D’autre part pour tout $\lambda > 0$ tel que $(\pi/L)^2 < \lambda < 4(\pi/L)^2$, il existe des solutions $u$ de l’équation de la chaleur à une dimension $u_t - u_{xx} + u^3 - \lambda u = 0$ dans $\mathbb{R}^+ \times (0, L)$, $u(t, 0) = u(t, L) = 0$ sur $\mathbb{R}^+$ telles que $\int_\Omega u(0, x) u(t, x) \, dx < 0$ pour un certain $t > 0$.

1 – A simple positivity property

Let $\Omega$ be a bounded, open subset of $\mathbb{R}^N$ with a Lipschitz continuous boundary and let us consider the linear parabolic equation

\begin{equation}
(1.1) \quad u_t - \Delta u + a(t, x) u = 0 \quad \text{in } \mathbb{R}^+ \times \Omega, \quad u = 0 \quad \text{on } \mathbb{R}^+ \times \partial \Omega,
\end{equation}

where $a \in L^\infty(\mathbb{R}^+ \times \Omega)$. For any $u_0 \in L^\infty(\Omega)$, there is a unique global solution

$$
u \in C\left([0, \infty); L^\infty(\Omega)\right) \cap C\left((0, \infty); H_0^1(\Omega)\right)$$

of (1.1) with initial datum $u(0, x) = u_0(x)$. It is well-known that (1.1) is positivity preserving in the sense that if $u_0 \geq 0$, then $u(t, x) \geq 0$ a.e. on $\mathbb{R}^+ \times \Omega$. For more

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general initial data, when \( a = 0 \) we know that the inner product \((u_0, u(t, \cdot))\) in the sense of \(L^2(\Omega)\) is nonnegative (in fact, even positive if \( u_0 \neq 0 \)) since the heat semi-group is the exponential of a self-adjoint operator. More generally we have the following

**Proposition 1.1.** Let \( L = L^* \) be a self-adjoint linear operator on a real Hilbert space \( H \), generating a strongly continuous semi-group on \( H \) and \( \alpha \in L^\infty(\mathbb{R}^+) \). Then for any \( u_0 \in H \), the unique mild solution \( u \in C([0, \infty); H) \) of

\[
(1.2) \quad u' = Lu + \alpha(t) u
\]

such that \( u(0) = u_0 \) is such that \((u_0, u(t)) \geq 0\) for all \( t \geq 0 \).

**Proof:** Denoting by \( A(t) \) the primitive of \( \alpha(t) \) which vanishes at 0 we have

\[
u(t) = \exp(A(t)) \exp(tL) u_0 \quad \text{for all} \quad t \geq 0.
\]

The result follows immediately since \( \exp(tL) = \exp[(t/2)L] \exp[(t/2)L]^* \geq 0 \).

**Corollary 1.2.** If \( a(t, x) = a_1(t) + a_2(x) \) with \( a_1 \in L^\infty(\mathbb{R}^+) \) and \( a_2 \in L^\infty(\Omega) \), then for any \( u_0 \in L^\infty(\Omega) \), the unique global solution \( u \) of (1.1) with initial datum \( u(0, x) = u_0(x) \) is such that \( u(0) = u_0 \) is such that \((u_0, u(t))_H \geq 0\) for all \( t \geq 0 \), where \((\ , \ )_H\) denotes the inner product in \( H = L^2(\Omega) \).

**Proof:** Just apply Proposition 1.2 with \( L = \Delta - a_2(x)I \) with Dirichlet boundary conditions.

### 2 – A counterexample

In the investigation of uniqueness of anti-periodic solutions to semi-linear parabolic equations (cf. e.g. [2, 5, 7, 8]) the question naturally arises of whether an equation such as (1.1) can have a non-trivial solution \( u \) with \( u(\tau, \cdot) = -u(0, \cdot) \) for some \( \tau > 0 \). Such a possibility would be excluded if we knew that Corollary 1.2 is valid for any potential \( \alpha \in L^\infty(\mathbb{R}^+ \times \Omega) \). As we shall see now, it is not the case. Consider the one-dimensional semilinear heat equation

\[
(2.1) \quad u_t - u_{xx} + cu^3 - \lambda u = 0 \quad \text{in} \quad \mathbb{R}^+ \times (0, L), \quad u(t, 0) = u(t, L) = 0 \quad \text{on} \quad \mathbb{R}^+, \quad c > 0, \lambda > 0.
\]

All solutions of this problem are global and uniformly bounded on \( \mathbb{R}^+ \times (0, L) \). For \( (\pi/L)^2 = \lambda_1(0, L) < \lambda < \lambda_2(0, L) = 4(\pi/L)^2 \), the stationary “elliptic problem”

\[
(2.2) \quad \varphi \in H_0^1(0, L), \quad -\varphi_{xx} + c \varphi^3 - \lambda \varphi = 0
\]

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has exactly 3 solutions, namely 0, the positive solution $\varphi$ and the negative solution $(-\varphi)$. Setting $\Omega = (0, L)$, we shall establish

**Theorem 2.1.** For any $c > 0$ and $\lambda_1 < \lambda < \lambda_2$, there is $u_0 \in L^\infty(\Omega)$ such that the unique global solution $u$ of (2.1) with initial datum $u(0, x) = u_0(x)$ satisfies

$$
(2.3) \quad \int_\Omega u_0(x) u(t, x) \, dx < 0
$$

for some $t > 0$.

**Proof:** We proceed by contradiction. Assume, instead of (2.3), that for all $u_0 \in L^\infty(\Omega)$ we have

$$
(2.4) \quad \forall t \geq 0, \quad \int_\Omega u_0(x) u(t, x) \, dx \geq 0.
$$

Since any solution $u$ of (2.1) is well-known (cf. e.g. [4]) to converge at infinity to one of the 3 solutions of (2.2), let us investigate first what happens if $u(t, \cdot)$ converges to $\varphi$ as $t \to \infty$. From (2.4) we deduce immediately, by passing to the limit

$$
(2.5) \quad \int_\Omega u_0(x) \varphi(x) \, dx \geq 0.
$$

At this stage, changing if necessary $u$ to $(-u)$, we have obtained the following properties:

- If $u(t, \cdot)$ converges to $\varphi$ as $t \to \infty$, then $\int_\Omega u_0(x) \varphi(x) \, dx \geq 0$.
- Similarly if $u(t, \cdot)$ converges to $(-\varphi)$ as $t \to \infty$, then $\int_\Omega u_0(x) \varphi(x) \, dx \leq 0$.

To derive a contradiction, we shall prove the following

**Lemma 2.2.** Assuming $\int_\Omega u_0(x) \varphi(x) \, dx > 0$, we have $u(t) \to \varphi$ as $t \to \infty$, and

$$
(2.6) \quad \forall t \geq 0, \quad \int_\Omega u(t, x) \varphi(x) \, dx \geq 0.
$$

**Proof:** Since by the previous results $u(t)$ cannot tend to $(-\varphi)$ as $t \to \infty$, we must have either $u(t) \to 0$ or $u(t) \to \varphi$ as $t \to \infty$. Now let $u_\varepsilon$ be the solution of equation (2.1) such that $u_\varepsilon(0) = u_0 - \varepsilon \varphi$ with $\varepsilon > 0$. For $\varepsilon > 0$ small enough, we have $\int_\Omega (u_0(x) - \varepsilon \varphi(x)) \varphi(x) \, dx > 0$, and the solution $u_\varepsilon$ of equation (2.1) such that $u_\varepsilon(0) = u_0 - \varepsilon \varphi$ also tends either to 0 or $\varphi$ at infinity while $w := u - u_\varepsilon \geq 0$. 

Now if \( u(t) \to 0 \) as \( t \to \infty \), we also must have \( u_\varepsilon(t) \) as \( t \to \infty \), both convergences being uniform on \([0, L]\). Since \( \lambda > \lambda_1(0, L) = (\pi/L)^2 \), an immediate calculation now shows that, as a consequence of the equation

\[
w_t - w_{xx} + c(u^2 + u_\varepsilon + u_\varepsilon^2) w = \lambda w \quad \text{in} \quad \mathbb{R}^+ \times (0, L), \quad w(t, 0) = w(t, L) = 0 \quad \text{on} \quad \mathbb{R}^+ \n
\]

there exists \( T > 0 \) and \( \eta > 0 \) for which

\[
\forall t \geq T, \quad \frac{d}{dt} \int_\Omega w(t, x) \psi(x) \, dx \geq \eta \int_\Omega w(t, x) \psi(x) \, dx
\]

with \( \psi(x) := \sin(\pi/L) x \) on \([0, L]\). Of course this implies that either \( w = 0 \) for \( t \geq T \), excluded by backward uniqueness (cf. e.g. [1, 3]) or \( w \) is unbounded as \( t \to \infty \), a contradiction. Consequently we must have \( u(t) \to \varphi \) as \( t \to \infty \). Then (2.6) follows from the fact that for each \( \tau > 0 \), \( v(t, \cdot) = u(t+\tau, \cdot) \) is a solution of (2.1) with \( v(t) \to \varphi \) as \( t \to \infty \).

**Proof of Theorem 2.1 (continued):** We now turn our attention to those initial data \( u_0 \) orthogonal to \( \varphi \) in \( H \), which means

\[
(2.7) \quad \int_\Omega u_0(x) \varphi(x) \, dx = 0.
\]

Considering \( v_\varepsilon \) the solution of equation (2.1) such that \( v_\varepsilon(0) = u_0 + \varepsilon \varphi \) with \( \varepsilon > 0 \), we remark that as \( \varepsilon \to 0 \), \( v_\varepsilon(t, \cdot) \) converges to \( u(t, \cdot) \) uniformly for each \( t \geq 0 \) fixed. By Lemma 2.2 we have

\[
\forall t \geq 0, \quad \int_\Omega v_\varepsilon(t, x) \varphi(x) \, dx \geq 0
\]

and by letting \( \varepsilon \to 0 \), we deduce:

\[
\forall t \geq 0, \quad \int_\Omega u(t, x) \varphi(x) \, dx \geq 0.
\]

Changing \( u_0 \) to \((-u_0)\), from (2.7) we also deduce

\[
\forall t \geq 0, \quad \int_\Omega u(t, x) \varphi(x) \, dx \leq 0.
\]

Hence finally (2.7) implies

\[
(2.8) \quad \forall t \geq 0, \quad \int_\Omega u(t, x) \varphi(x) \, dx = 0.
\]
The fact that (2.7) implies (2.8) is contradictory with direct properties of (2.1).

Since \( \varphi(x) \) is not constant, there is \( h(x) \in L^2(\Omega) \) such that, for instance

\[
(2.9) \quad \int_{\Omega} h(x) \varphi(x) \, dx = 0, \quad \int_{\Omega} h(x) \varphi^3(x) \, dx > 0.
\]

Let \( h_n(x) \) be a sequence of \( C^\infty \) functions with compact support converging to \( h \) in \( L^2(\Omega) \). For \( n \) large we have

\[
\frac{\int_{\Omega} h_n(x) \varphi(x) \, dx}{\int_{\Omega} \varphi^2(x) \, dx} = c_n \to 0
\]

while \( \int_{\Omega} (h_n(x) - c_n \varphi(x)) \varphi(x) \, dx = 0 \) and \( \int_{\Omega} (h_n(x) - c_n \varphi(x)) \varphi^3(x) \, dx > 0 \) for \( n \) large. Therefore we can find \( h(x) \in L^\infty(\Omega) \) (and even a \( C^\infty \) function) satisfying (2.9). Picking \( u_0 = \alpha h \) with \( \alpha > 0 \) small enough, we now find

\[
(2.10) \quad \int_{\Omega} u_0(x) \varphi(x) \, dx = 0, \quad \int_{\Omega} u_0(x) \varphi(x) \left( \varphi^3(x) - u_0^2(x) \right) \, dx > 0.
\]

On the other hand for \( t > 0 \) we have

\[
\frac{d}{dt} \int_{\Omega} u(t,x) \varphi(x) \, dx = \int_{\Omega} u_t(t,x) \varphi(x) \, dx = \int_{\Omega} \left( u_{xx} - u^3 + \lambda u \right) \varphi \, dx
\]

\[
= \int_{\Omega} \varphi_{xx} \lambda \varphi \, u \, dx - \int_{\Omega} u^3 \, \varphi \, dx = \int_{\Omega} u \varphi (\varphi^2 - u^2) \, dx.
\]

By considering small values of \( t \), we see that \( \int_{\Omega} u(t,x) \varphi(x) \, dx \) is increasing on a small time interval \([t', t'']\). Since \( \int_{\Omega} u_0(x) \varphi(x) \, dx = 0 \), this contradicts (2.8). The proof of Theorem 2.1 is now complete.

**Corollary 2.3.** The conclusion of Corollary 1.2 is not valid for a general potential \( a \in (\mathbb{R}^+ \times \Omega) \).

**Proof:** We choose \( \Omega = (0, L) \), \( u_0 \in L^\infty(\Omega) \) such that the unique global solution \( u \) of (2.1) with initial datum \( u(0, x) = u_0(x) \) satisfies (2.3), and \( a(t, x) := cu^2 - \lambda \).

**REFERENCES**


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