PORTUGALIAE MATHEMATICA
Vol. 54 Fasc. 2 – 1997

ON PARA-KÄHLERIAN MANIFOLDS $M(J,g)$ AND ON SKEW SYMMETRIC KILLING VECTOR FIELDS CARRIED BY M

I. Mihai, L. Nicolescu and R. Rosca

Abstract: Para-complex manifolds and, in particular, para-Kählerian manifolds have been for the first time studied by Rashevski [Ra], Libermann [L] and Patterson [Pa]. In the last two decades, several authors have dealt with such type of manifolds, as for instance [R1], [R2], [Cr], [GM], [RMG], [CFG] and some others. A para-Kählerian manifold is a manifold endowed with an almost product structure (called also a para-complex structure) J and a pseudo-Riemannian metric g, which satisfy the conditions of compatibility $g\circ(J\times J)=-g$ and $\nabla J=0$, where ∇ is the Levi–Civita connection with respect to g.

In the present paper, adopting P. Libermann stand point, we study some properties of a para-Kählerian manifold M and emphasize the case when M carries a non null skew symmetric Killing vector field (in the sense of R. Rosca [R4], [R5]).

1 – Preliminaries

A *para-Kählerian* manifold is a manifold M endowed with an almost product structure J (i.e. an involutive endomorphism of TM) and a pseudo-Riemannian metric g, which satisfy the conditions of compatibility

$$g(JZ,Z')+g(Z,JZ')=0\,,\qquad \nabla J=0\,, \tag{1.1}$$

where ∇ is the Levi–Civita connection with respect to g. From these conditions, it follows that $\dim M=2m$, g is neutral and

$$Tr\,J=0,\quad N_J=0,\quad \nabla\Omega=0,\quad d\Omega=0\,, \tag{1.2}$$

where Ω is the symplectic form determined by J and g, i.e. $\Omega(Z,Z')=g(JZ,Z')$, $\forall\,Z,Z'\in\Gamma TM$ (ΓTM denotes the set of sections of the tangent bundle TM).

Received: February 19, 1996; *Revised*: July 11, 1996.

Following [P], we set:

$$A^q(M,TM) = \mathrm{Hom}\Big(\bigwedge^q TM, TM\Big)$$

and notice that elements of $A^q(M,TM)$ are vector valued q-forms (called also TM-valued forms). Denote by $\flat\colon TM \to T^*M$ the *g-musical* isomorphism (i.e. the canonical isomorphism defined by g) and by $d^\nabla\colon A^q(M,TM) \to A^{q+1}(M,TM)$ the *exterior covariant derivative* operator with respect to ∇. It should be noticed that generally $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d \circ d = 0$. If we denote by $p \in M$ the generic element of M, then the canonical TM-valued 1-form $dp \in A^1(M,TM)$ is also called the *soldering* form of M. Since ∇ is symmetric, one has $d^\nabla(dp) = 0$.

The operator $d^\omega = d + e(\omega)$ acting on $\bigwedge M$ is called the *cohomology operator* [GL]; $e(\omega)$ means the exterior product by the closed 1-form $\omega \in \bigwedge^1 M$, i.e. $d^\omega u = du + \omega \wedge u$, for any $u \in \bigwedge M$ ($d^\omega \circ d^\omega = 0$). If $d^\omega u = 0$, it is said that u is d^ω-closed and if ω is exact, then u is said to be a *d^ω-exact form*.

Any vector field $Z \in \Gamma TM$ such that

$$d^\nabla(\nabla Z) = \nabla^2 Z = \sigma \wedge dp \;\in\; A^2(M,TM)\ ,$$

for some 1-form σ is said to be an *exterior concurrent* vector field [PRV]. The 1-form σ is called the concurrence form and is expressed by $\sigma = f\flat(X)$, $f \in C^\infty M$.

One may consider on M a field of adapted Witt vector frames

$$W = \mathrm{vect}\Big\{h_a, h_{a^*} \mid a \in \{1,\ldots,m\};\ a^* = a + m\Big\}\ ,$$

where h are null real vector fields which satisfy $g(h_a, h_{a^*}) = 1$ and all the other products are 0. With respect to the operator J, the vector fields h satisfy

$$Jh_a = h_a\,, \qquad Jh_{a^*} = -h_{a^*}\ , \tag{1.3}$$

and the above relations define a J-null vector basis on M.

If $W^* = \mathrm{covect}\{\omega^a, \omega^{a^*}\}$ is the associated cobasis of W, then the soldering form dp, the structure 2-form Ω and the metric tensor g are expressed by

$$dp = \omega^A \otimes h_A\,, \qquad A \in \{a, a^*\}\ , \tag{1.4}$$

$$\Omega = \Sigma\omega^a \wedge \omega^{a^*}\ , \tag{1.5}$$

$$g = \langle dp, dp\rangle = \Sigma\omega^a\,\omega^{a^*}\ , \tag{1.6}$$

which shows that the para-Hermitian metric g is exchangeable with Ω.

Let now $\theta^A_B \in \bigwedge^1 M$ (resp. $\Theta^A_B \in \bigwedge^2 M$) be the local connection forms in the tangent bundle TM (resp. the curvature 2-forms in TM). Then the structure equations (E. Cartan) may be written in indexless form as:

$$\nabla h = \theta \otimes h \ \in \ A^1(M, TM) \ , \tag{1.7}$$

$$d\omega = -\theta \wedge \omega \ , \tag{1.8}$$

$$d\theta = -\theta \wedge \theta + \Theta \ . \tag{1.9}$$

By (1.1), (1.2) and (1.7) the connection forms θ satisfy

$$\theta^a_b + \theta^{b^*}_{a^*} = 0 \, , \quad \theta^{a^*}_b = 0 \, , \quad \theta^a_{b^*} = 0 \ , \tag{1.10}$$

which shows that the connection matrix $\mathcal{M}_\theta$ is the Chern–Libermann matrix

$$\mathcal{M}_\theta = \begin{pmatrix} \theta^a_b & 0 \\ 0 & \theta^{a^*}_{b^*} \end{pmatrix} \ . \tag{1.11}$$

Further by (1.10) and (1.9) one has

$$\Theta^a_b + \Theta^{b^*}_{a^*} = 0 \, , \quad \Theta^{a^*}_b = 0 \, , \quad \Theta^a_{b^*} = 0 \ . \tag{1.12}$$

We also recall that

$$\theta_R = \Sigma \theta^a_a = -\Sigma \theta^{a^*}_{a^*} \ , \tag{1.13}$$

$$\Theta_R = d\theta_R = \Sigma \Theta^a_a = -\Sigma \Theta^{a^*}_{a^*} \tag{1.14}$$

are called the Ricci 1-form and the Ricci 2-form of M, respectively.

Denote now by

$$S_p = \operatorname{span}\{h_a\}_p, \quad S^*_p = \operatorname{span}\{h_{a^*}\}_p \, , \quad \forall\, p \in M \ . \tag{1.15}$$

Then S and S^* defines two self orthogonal distributions associated with the J-null vector basis $W = \operatorname{vect}\{h_A\}$.

If T_pM is the tangent space of M at $\forall\, p \in M$, one has the standard decomposition [L], [R1]

$$T_pM = S_p \oplus S^*_p \tag{1.16}$$

and

$$JS_p = S_p \, , \quad JS^*_p = S^*_p \ . \tag{1.17}$$

Next denote by

$$\psi = \omega^1 \wedge ... \wedge \omega^m \ , \tag{1.18}$$

$$\psi^* = \omega^{m+1} \wedge ... \wedge \omega^{2m} \tag{1.19}$$

the simple unit m-forms which correspond to S and S^* respectively. By (1.5), (1.10) and (1.13) exterior differentiation of (1.18) and (1.19) gives

$$d\psi = -\theta_R \wedge \psi \ , \tag{1.20}$$

$$d\psi^* = \theta_R \wedge \psi^* \ . \tag{1.21}$$

The above equations show that both m-forms ψ and ψ^* are exterior recurrent [D] and have $-\theta_R$ and θ_R respectively as recurrence 1-forms. Hence ψ and ψ^* are locally completely integrable.

Further since ψ annihilates S^* and ψ^* annihilates S, it follows from (1.20), (1.21) and by Frobenius theorem that both distributions S and S^* are *involutive*.

It is worth to notice that in this situation $-\theta_R$ (resp. θ_R) is an element of the first class of cohomology $H^1(S^*, \mathbf{R})$ (resp. of $H^1(S, \mathbf{R})$).

Moreover, since

$$(S_p)^\perp = S_p, \quad (S_p^*)^\perp = S_p^*, \quad \Omega|_{S_p} = 0, \quad \Omega|_{S_p^*} = 0 \ ,$$

it is seen that S and S^* are two Lagrangian *polarizations* of the symplectic structure defined by (W, Ω). We recall that in [R2], S and S^* have been defined as the natural polarizations of $M(J, g)$.

We conclude this section with the following meaningful remark. Consider the TM-valued 1-form

$$Jdp = \omega^a \otimes h_a - \omega^{a^*} \otimes h_{a^*} \ . \tag{1.22}$$

Clearly by (1.3) we have $\langle dp, Jdp \rangle = 0$ and $\langle dp, dp \rangle = -\langle Jdp, Jdp \rangle = g$.

Therefore one has the following.

Proposition. *To any para-Kählerian manifold $M(J, g)$ of soldering form dp and metric tensor g corresponds by orthogonality of line elements a para-Kählerian manifold of soldering form Jdp and metric tensor $-g$.*

2 – Principal Lagrangian submanifolds

Let now $Y^* \in S^*$ be any vector field of the self orthogonal distribution S^*. Since $i_{Y^*}\psi = 0$, then by (1.20) it quickly follows:

$$\mathcal{L}_{Y^*}\psi = -\theta_R(Y^*)\,\psi \tag{2.1}$$

($\mathcal{L}$: Lie derivative), that is Y^* defines an *infinitesimal conformal transformation* of ψ.

In similar manner, for all $Y \in S$ one gets by (1.21)

$$\mathcal{L}_{Y}\psi^* = \theta_R(Y)\,\psi^* \ . \tag{2.2}$$

Therefore one may say that the m-form ψ (resp. the m-form ψ^*) is S^**-conformal invariant* (resp. S*-conformal invariant*).

Further denote by $X \in \Gamma TM$ any vector field which annihilates the Ricci 1-form θ_R. Making use of (1.14) one derives from (2.1) and (2.2)

$$d(\mathcal{L}_X\psi) = -\theta_R \wedge \mathcal{L}_X\psi + \Theta_R \wedge i_X\psi \ , \tag{2.3}$$

$$d(\mathcal{L}_X\psi^*) = \theta_R \wedge \mathcal{L}_X\psi^* - \Theta_R \wedge i_X\psi^* \ . \tag{2.4}$$

By reference to [R3], [Pap], the above equations show that the Lie derivatives with respect to X of both m-forms ψ, ψ^* are *exterior quasi recurrent* and have $\Theta_R \wedge i_X\psi$ and $-\Theta_R \wedge i_X\psi^*$ as *exterior recurrence difference* respectively.

Now, in consequence of the splitting (1.16) and of (1.4), we set

$$dp_S = \omega^a \otimes h_a \,, \qquad dp_{S^*} = \omega^{a^*} \otimes h_{a^*} \ , \tag{2.5}$$

for the line elements of the principal Lagrangian foliations S and S^* on $M(J,g)$.

Operating on dp_S and dp_{S^*} by the covariant differential operator d^∇, one easily gets

$$d^\nabla(dp_S) = 0 \,, \qquad d^\nabla(dp_{S^*}) = 0 \ .$$

This shows the significant fact that dp_S (resp. dp_{S^*}) is the soldering form of the leaf M_S of S through p (resp. the leaf M_{S^*} of S^* through p).

Next we denote like usual by $*$ the Hodge star operator and recall that on an m-dimensional oriented manifold M, $*$ maps scalar or TM-valued q-forms into scalar or TM-valued $(n-q)$-forms.

Comming back to the case under discussion, one gets by (1.2) and (2.5)

$$*dp_S = \Sigma(-1)^a\,\omega^1 \wedge ... \wedge \widehat{\omega}^a \wedge ... \wedge \omega^m \otimes h_a \tag{2.6}$$

(the roof $\widehat{}$ indicates the missing term).

Operating on (2.6) by d^∇, the above relation moves after some calculations to

$$d^\nabla(*dp_S) = -\theta_R \wedge *dp_S \ . \tag{2.7}$$

Hence $*dp_S$ is an exterior recurrent TM-valued form, having $-\theta_R$ as recurrence 1-form. Therefore by reference to [Pap] one may say that line element dp_S is *exterior co-recurrent*.

In similar manner, one finds that the same property holds good for the line element dp_{S^*}, but with θ_R as recurrence form.

Since dp_S and dp_{S^*} are dual TM-valued forms, it follows, according to a known theorem, that the necessary and sufficient condition in order that dp_S and dp_{S^*} be *harmonic* TM-valued forms on $M(J,g)$ is that the Ricci 1-form vanishes. In this case following [Cru], it is proved that $M(J,g)$ is equipped with a *spin Euclidean connection*.

Let denote by $T^\perp_{p_S}(M_S)$ (resp. $T^\perp_{p_{S^*}}(M_{S^*})$) the normal space of M_S at $p_S \in M_S$ (resp. of M_{S^*} at $p_{S^*} \in M_{S^*}$). By (1.17), one has

$$JT_{p_S}(M_S) = T^\perp_{p_S}(M_S)\,, \qquad JT_{p_{S^*}}(M_{S^*}) = T^\perp_{p_{S^*}}(M_{S^*}) \ . \tag{2.8}$$

The above prove that with respect to the para-complex operator J, M_S and M_{S^*} are *anti-invariant* submanifolds [YK] of M.

In addition, if M admits a spin Euclidean connection (i.e. $\theta_R = 0$), we have seen that the soldering forms of M_S and M_{S^*} are *harmonic*. Then according to the improper immersions theory, we agree to say that M_S and M_{S^*} are improper minimal submanifolds of $M(J,g)$.

Moreover, since in the actual discussion $\theta_R = 0$, it follows by (1.20) and (1.21) that both simple unit m-forms ψ and ψ^* are *harmonic*. Then since $M_S \cap M_{S^*} = \{0\}$, we are in the condition of Tachibana's theorem [T] in case of proper immersions. As a consequence of this fact, we may say that $M(J,g)$ is the local product

$$M = M_S \times M_{S^*} \ ,$$

where M_S and M_{S^*} are anti-invariant and improper minimal submanifolds of $M(J,g)$.

Summarizing, we proved the following.

Theorem. *Let M_S and M_{S^*} be the two principal Lagrangian submanifolds of a $2m$-dimensional para-Kählerian manifold $M(J,g)$. Let θ_R be the Ricci 1-form on M, dp_S and dp_{S^*} the soldering forms of M_S and M_{S^*} and ψ (resp. ψ^*) the volume element of M_S (resp. M_{S^*}). One has the following properties:*

i) *ψ (resp. ψ^*) is S^*-conformal invariant (resp. S-conformal invariant);*

ii) *if X is any vector field of M which annihilates θ_R, then the Lie derivatives of ψ and ψ^* with respect to X are quasi recurrent;*

iii) *dp_S and dp_{S^*} are exterior co-recurrent with $-\theta_R$ and θ_R respectively as recurrence forms;*

iv) *if M admits a spin Euclidean connection (i.e. $\theta_R = 0$), then M may be viewed as the local product*

$$M = M_S \times M_{S^*} \ ,$$

such that M_S and M_{S^*} are both improper minimal and anti-invariant submanifolds of M.

3 – J-skew symmetric Killing vector fields

We assume in this Section that a para-Kählerian manifold $M(J,g)$ carries a J-skew symmetric Killing vector field X. Then, following R. Rosca [R4] (see also [DRV], [MMR]) such a vector field is defined by

$$\nabla X = X \wedge JX \quad \Leftrightarrow \quad \nabla X = \flat(JX) \otimes X - \flat(X) \otimes JX \ , \tag{3.1}$$

where $\wedge$ means the *wedge product* of vector fields ($\wedge(X,\cdot)$ is a linear operator which is skew symmetric with respect to $\langle\ ,\ \rangle$, $\forall\, X \in \Gamma\, TM$).

In order to simplify, we write

$$\alpha = \flat(X)\,, \qquad \beta = \flat(JX)\,, \qquad 2l = \|X\|^2 \ . \tag{3.2}$$

Setting

$$X = \Sigma(X^a h_a + X^{a^*} h_{a^*}) \ , \tag{3.3}$$

then since $W = \{h\}$ is a Witt vector basis, it follows that

$$\alpha = \Sigma(X^a \omega^{a^*} + X^{a^*} \omega^a) \ , \tag{3.4}$$

$$\beta = \Sigma(X^a \omega^{a^*} - X^{a^*} \omega^a) \ . \tag{3.5}$$

In these conditions it quickly follows from (3.1)

$$dl = 2l\beta \quad \Rightarrow \quad d\beta = 0 \ , \tag{3.6}$$

which shows that JX is a gradient vector field.

Next making use of the structure equations (1.7) and taking account of (3.3) one derives

$$d\alpha = 2\beta \wedge \alpha \quad \Leftrightarrow \quad d^{-2\beta}\alpha = 0 \; . \tag{3.7}$$

Now since β is an exact form, the above equation shows that in terms of d^{ω}-cohomology α is a $d^{-2\beta}$-*exact form*. On the other hand, since $(\nabla J)Z = 0$ and $J^2 = Id$, one gets from (3.1)

$$\nabla JX = \beta \otimes JX - \alpha \otimes X \; . \tag{3.8}$$

But since by (1.1) one has $\alpha(JX) = 0$, it follows from (3.8)

$$\nabla_{JX} JX = -\|X\|^2 \, JX \,, \qquad \beta(JX) = -\|X\|^2 \tag{3.9}$$

and by (3.1) one finds at once $[X, JX] = 0$. Then we may say that JX is *affine geodesic* which commutes with X. One also checks the Ricci identity

$$\mathcal{L}_Z g(X, JX) = g(\nabla_Z X, JX) + g(X, \nabla_Z JX) \,, \qquad Z \in \Gamma TM \, .$$

Let now Σ be the exterior differential system which defines the skew symmetric Killing vector field X. By (3.6) and (3.7) it is seen that the *characteristic numbers* of Σ are $r = 3$, $s_0 = 1$, $s_1 = 2$. Since $r = s_0 + s_1$, Σ is in involution in the sense of E. Cartan [C] and we may say that the existence of X depends on 2 arbitrary functions of 1 argument.

We denote by $D_X = \{X, JX\}$ the holomorphic distribution [GM] defined by X and JX. Then on behalf of (3.1) and (3.3), if X', X'' are any vector fields of D_X one has $\nabla_{X'} X'' \in D_X$. This as is known proves that D_X is an *autoparallel foliation* and that its leaves M_X are totally geodesic surfaces of M. Then by Frobenius' theorem put $M_X^{\perp}$ the 2-codimensional submanifold orthogonal to M_X.

Since $M_X^{\perp}$ is defined by $\alpha = 0$, $\beta = 0$, it is seen by (3.1), (3.8) that X and JX are geodesic normal sections of $M_X^{\perp}$. Therefore we conclude by the following significative fact: any para-Kählerian manifold M which carries a J-skew symmetric Killing vector field X may be viewed as the local Riemannian product

$$M = M_X \times M_X^{\perp} \; ,$$

such that:

i) M_X is a para-holomorphic totally geodesic surface tangent to X and JX;

ii) $M_X^{\perp}$ is a 2-codimensional totally geodesic submanifold of M.

By (3.1) and (3.8) one has

$$\nabla_{JX} X + \nabla_X JX = -2\,\|X\|^2\,X$$

and by (3.9) it is easily seen that the conditions:

i) X is a null vector field (see [DRV]);

ii) JX is a geodesic;

iii) X and JX are left invariant;

are mutually equivalent.

Now following [KN], if we set

$$A_X X = -\nabla_X X = \|X\|^2\,JX\ , \tag{3.10}$$

one checks the general formula

$$\frac{1}{2}\,\mathcal{L}_Z\,\|X\|^2 = g(Z, A_X X)\,, \qquad Z \in \Gamma TM\,.$$

Next since $\operatorname{div} JX = \operatorname{tr}[\nabla JX]$ (Hermitian trace understood) one gets by an easy calculation $\operatorname{div} JX = -2\|X\|^2$, and taking account of (3.6) one gets

$$\operatorname{div} A_X X = -8l\|X\|^2 = -4\|X\|^2\ . \tag{3.11}$$

Next by (3.6) one may write

$$\nabla 2l = \nabla\|X\|^2 = 2\|X\|^2\,JX$$

(∇ denotes the gradient of a scalar) and by (3.11) one finds

$$\operatorname{div}\nabla\|X\|^2 = -8\|X\|^4\ . \tag{3.12}$$

Since one has $\|\nabla\|X\|^2\|^2 = -4\|X\|^6$, it is seen by (3.12) that $\|\nabla\|X\|^2\|^2$ and $\operatorname{div}\nabla\|X\|^2$ are functions of $\|X\|^2$. Hence following a known definition (see also [W]), $\|X\|^2$ is an *isoparametric function*.

Further with respect to the Witt basis one has

$$\|\nabla X\|^2 = 2g(\nabla_{h_a}X, \nabla_{h_{a^*}}X) = 2\|X\|^4\ . \tag{3.13}$$

But if $Z \in \Gamma TM$ one has as is known [P], $\Delta\|Z\|^2 = -\operatorname{div}\nabla\|Z\|^2$ and in consequence of (3.13) one gets

$$\Delta\|X\|^2 = 8\|X\|^4\ . \tag{3.14}$$

Now making use of the general Bochner formula

$$2\langle \mathrm{tr}\,\nabla^2 X, X\rangle + 2\|\nabla X\|^2 + \Delta\|X\|^2 = 0$$

(for any vector field X), one finds by (3.13) and (3.14)

$$\langle \mathrm{tr}\,\nabla^2 X, \cdot\rangle + 6\|X\|^4 = 0\ . \tag{3.15}$$

Recalling now that for any Killing vector field X one has

$$\langle \mathrm{tr}\,\nabla^2 X, \cdot\rangle + \mathcal{R}(X, \cdot) = 0$$

($\mathcal{R}$ denotes the Ricci tensor field of ∇), one derives at once from (3.15), that in the case under discussion one has

$$\mathcal{R}(X, X) = 6\|X\|^4 \quad\Rightarrow\quad \mathrm{Ric}(X) = 6\|X\|^2$$

($\mathrm{Ric}(X)$ is the Ricci curvature of M with respect to X).

We associate to X and JX the following two null vector fields:

$$Y = X + JX \;\in\; S\,, \qquad Y^* = X - JX \;\in\; S^*\ . \tag{3.16}$$

By (3.1) and (3.3) one gets at once

$$\nabla Y = (\beta - \alpha)\otimes Y\,, \qquad \nabla Y^* = (\beta + \alpha)\otimes Y^*\ , \tag{3.17}$$

which show that Y and Y^* are recurrent vector fields. It should be noticed, since Y and Y^* are null vector fields, that they are also defined as geodesic vectors.

Moreover operating on (3.17) by d^∇ a short calculation gives

$$d^\nabla(\nabla Y) = \nabla^2 Y = 2(\alpha\wedge\beta)\,Y\ ,$$
$$d^\nabla(\nabla Y^*) = \nabla^2 Y^* = -2(\alpha\wedge\beta)\,Y^*\ ,$$

which defines Y and Y^* also as *2-recurrent vector fields* [EM].

We remark that the wedging of Y and Y^* is (up to 2) the covariant derivative of JX, i.e.

$$\nabla JX = \frac{1}{2}(Y\wedge Y^*)\ .$$

Next operating on (3.1) and (3.8) by d^∇, one derives

$$\nabla^2 X = 2(\alpha\wedge\beta)\otimes JX\,, \qquad \nabla^2 JX = 2(\alpha\wedge\beta)\otimes X\ . \tag{3.18}$$

So, if we set $F = \nabla X$, we get

$$d^{\nabla^2} F = \nabla^2 X \wedge \beta - \nabla^2 JX \wedge \alpha = 0 \ .$$

Hence the d^{∇^2}-differential of F is closed; it follows that $\nabla^3 X$ is a 0-*element* of $A^3(M, TM)$.

Since $g(X, JX) = 0$, the *para-holomorphic sectional curvature* $K_{X\wedge JX}$ defined by X is expressed by

$$K_{X\wedge JX} = \frac{g\Big(R(X,JX)\,JX,\ X\Big)}{\|X\|^2\,\|JX\|^2} \ ,$$

where R denotes the Riemannian curvature tensor. But

$$R(X,JX)\,JX = \nabla^2 JX(X,JX)$$

and making use of (3.18) one finds

$$K_{X\wedge JX} = \|X\|^2 = \frac{1}{6}\,\mathrm{Ric}(X) \ ,$$

which relates the para-holomorphic sectional curvature defined by X to $\mathrm{Ric}(X)$.

We will investigate some infinitesimal transformations of the Lie algebra $\bigwedge M$, induced by the vector fields X and JX. By (2.7) and (3.7) one gets

$$\mathcal{L}_X \alpha = 0\,, \quad \mathcal{L}_{JX} \beta = -2\|X\|^2\,\beta \ , \tag{3.19}$$

$$\mathcal{L}_X \beta = 0\,, \quad \mathcal{L}_{JX} \alpha = -2\|X\|^2 \alpha \ . \tag{3.20}$$

From above it is seen that α and β are *invariant* by X and that JX defines *infinitesimal conformal transformations* of α and β. In particular, X is a *self-invariant* vector field whilst JX is a *self-conformal* vector field.

We recall now that the bracket $[\ ,\ ]$ of two T^*M-valued forms $F = Z_i \omega^i$, $F' = Z'_i \omega'^i$ is defined by

$$[F, F'] = [Z_i, Z'_j]\,\omega^i \wedge \omega'^j$$

and the Lie derivative of $[\ ,\ ]$ with respect to a vector field U is given by

$$\mathcal{L}_U [F, F'] = [F, \mathcal{L}_U F'] + [\mathcal{L}_U F, F'] \ .$$

Comming back to the case under discussion and setting $F = \nabla X$, $F' = \nabla JX$ one gets by (3.19), (3.20)

$$\mathcal{L}_X [\nabla X, \nabla JX] = 0 \ , \tag{3.21}$$

$$\mathcal{L}_{JX} [\nabla X, \nabla JX] = -4\|X\|^2\,[\nabla X, \nabla JX] \ . \tag{3.22}$$

Hence the bracket $[\nabla X, \nabla JX]$ is invariant by X and JX defines an infinitesimal conformal transformation of $[\nabla X, \nabla JX]$ having $-4\|X\|^2$ as conformal scalar.

Next following [LM] we denote by

$$Z \mapsto -i_Z\Omega = \Omega^\flat(Z) =^\flat Z$$

the symplectic isomorphism defined by the structure form of $M(J,g)$.

Set $\mathcal{V}$ (resp. $\mathcal{V}^*$) for the symplectic vector space (resp. its dual). Then if ω is any form, its dual with respect to Ω is expressed by $\omega^\sharp\colon \mathcal{V}^* \to \mathcal{V}$.

Comming back to the case under discussion one quickly gets by (1.1) and (3.3)

$$\beta = \flat(JX) = -^\flat X \quad \Rightarrow \quad \mathcal{L}_X\Omega = 0\ . \tag{3.23}$$

But since β is an exact form, it follows according to a known definition that X is a global *Hamiltonian* vector field of Ω.

Further by (3.6) is proved the salient fact that $\frac{1}{2}\lg\frac{\|X\|^2}{2}$ is a *Hamiltonian function* of the symplectic form Ω. One may also say that X is a gradient of $\frac{1}{2}\lg\frac{\|X\|^2}{2}$.

Moreover consider the vector field

$$X_\nu = c\,JX + \nu X\,, \qquad c = \text{const.}, \quad \nu \in C^\infty M\ .$$

One has

$$^\flat X_\nu = -c\,\beta - \nu\,\alpha$$

and if ν satisfies

$$d\nu + 2\,\nu\,\beta = 0 \quad \Rightarrow \quad l\,\nu = \text{const.}\ ,$$

then $\mathcal{L}_{X_\nu}\Omega = 0$. Hence we may say that X_ν is a local Hamiltonian of Ω.

It should be noticed that if $\mathcal{E}^*_\Omega(M) = \{df^\sharp;\ f \in \bigwedge^0 M\}$ and $\mathcal{E}_\Omega(M) = \{\omega^\sharp;\ \omega \in \bigwedge^1 M\}$ denote the space of globally Hamiltonian vector fields and the space of local Hamiltonian vector fields, then as is known [G] one has $\mathcal{E}^*_\Omega(M) \subset \mathcal{E}_\Omega(M)$.

Therefore the following theorem is proved.

Theorem. *Let $M(J,g)$ be a para-Kählerian manifold. The existence on M of a J-skew symmetric Killing vector field X is determined by an exterior differential system in involution.*

Any M which carries such an X may be viewed as the local Riemannian product $M = M_X \times M_X^\perp$, such that:

1) *M_X is a totally geodesic para-holomorphic surface tangent to X and JX;*

2) *$M_X^\perp$ is a 2-codimensional totally geodesic submanifold of M.*

The following properties are induced by X:

i) *JX is an affine geodesic which commutes with X;*

ii) *$\|X\|^2$ is an isoparametric function and the Ricci curvature* $\mathrm{Ric}(X)$ *of M with respect to X and the para-holomorphic sectional curvature $K_{X\wedge JX}$ defined by X are related by*

$$\mathrm{Ric}(X) = 6\|X\|^2 = 6K_{X\wedge JX}\ ;$$

iii) *with X are associated two null (real) vector fields $Y = X + JX$, $Y^* = X - JX$ which enjoy the property to be 1-recurrent and 2-recurrent;*

iv) *the dual form $\flat(X) = \alpha$ (resp. $\flat(JX) = \beta$) of X (resp. JX) are invariant by X and JX defines an infinitesimal conformal transformation of both α and β, having $-2\|X\|^2$ as conformal scalar;*

v) *X is a global Hamiltonian vector field of Ω and any vector field $X_\nu = c\,JX + \nu X$ ($c =$ conts., $\nu \in C^\infty M$) such that $\nu\|X\|^2 =$ const. is a local Hamiltonian;*

vi) *the Lie bracket $[\nabla X, \nabla JX]$ is invariant by X, that is $\mathcal{L}_X[\nabla X, \nabla JX] = 0$.*

ACKNOWLEDGEMENT – The authors would like to thank the referee for valuable comments and suggestions.

REFERENCES

[C] Cartan, E. – *Systèmes Différentiels Extérieurs; Applications Géométriques*, Hermann, Paris, 1945.

[Cr] Cruceanu, V. – Une structure para-Kählerienne sur le fibré tangent, *Tensor N.S.,* 34 (1982), 81–84.

[CFG] Cruceanu, V., Fortuny, P. and Gadea, P.M. – *Para-Hermitian and para-Kähler manifolds*, preprint.

[Cru] Crumeyrolle, A. – Derivations, formes et opérateurs sur les champs spinoriels des variétés différentiables de dimension paire, *Ann. Inst. H. Poincaré,* 16 (1972), 182–190.

[D] Datta, D.K. – Exterior recurrent forms on a manifold, *Tensor N.S.,* 36 (1982), 115–120.

[DRV] Defever, F., Rosca, R. and Verstraelen, L. – On null Killing vector fields, *Riv. Mat. Univ. Parma,* 1(5) (1992), 239–253.

[EM] Enghis, P. and Margineanu, S. – Relations between Riemann spaces with birecurrent tensors*, Babes-Bolyai Univ. Math. Res. Sem.,* 2 (1988), 65–68.

[GM] Gadea, P.M. and Montesinos Amilibia, A. – Some geometric properties of para-Kählerian space forms, *Rend. Sem. Mat. Univ. Cagliari,* 59 (1989), 131–145.

[G] Grabowski, J. – *Derivations of the Lie algebras associated with a symplectic structure*, Differential Geometry and Its Applications; Proceedings, Brno, 1986.

[GL] Guedira, F. and Lichnerowicz, A. – Géométrie des algèbres de Lie locales de Kirilov, *J. Math. Pures Appl.,* 63 (1984), 407–484.

[KN] Kobayashi, S. and Nomizu, K. – *Foundations of Differential Geometry II*, Interscience, New York, 1969.

[L] Libermann, P. – Sur le problème d'équivalence de certaines structures infinitésimales, *Ann. di Mat.,* 36 (1954), 27–120.

[LM] Libermann, P. and Marle, C.M. – *Symplectic Geometry and Analytical Mechanics*, D. Reidel Publ. Co., Dordrecht/Boston, 1987.

[MMR] Matsumoto, K., Mihai, I. and Rosca, R. – A certain locally conformal almost cosymplectic manifold and its submanifolds, *Tensor N.S.,* 51 (1992), 91–102.

[Pap] Papadopoulos, A. – Exterior recurrent and co-recurrent forms on a manifold, *Tensor N.S.,* 40 (1983), 103–106.

[Pa] Patterson, F.M. – Riemannian extensions which have Kähler metrics, *Proc. Roy. Soc. Edinburgh,* 64 (1954), 113–126.

[PRV] Petrovic, M., Rosca, R. and Verstraelen, L. – Exterior concurrent vector fields on a Riemannian manifold, *Soochow J. Math.,* 15 (1989), 179–187.

[P] Poor, W.A. – *Differential Geometric Structures*, Mc Graw-Hill, New York, 1991.

[Ra] Rashevsky, K. – The scalar field in a stratified space, *Trudy Sem. Vekt. Tens. Anal.,* 6 (1968), 225–248.

[RMG] Reyes, E., Montesinos Amilibia, A. and Gadea, P.M. – Connections partially adapted to a metric ($J^4 = 1$)-structure, *Colloq. Math.,* 54 (1987), 216–229.

[R1] Rosca, R. – Para-Kählerian manifolds carrying a pair of concurrent self-orthogonal vector fields, *Abh. Math. Hamburg,* 46 (1977), 205–215.

[R2] Rosca, R. – CR-sous-variétés coisotropes d'une variété para-Kählerienne, *C.R. Acad. Sci. Paris,* 298 (1984), 149–151.

[R3] Rosca, R. – Variétés neutres $\widetilde{M}$ admettant une structure conforme symplectique et feuilletage co-isotrope, *C.R. Acad. Sci. Paris,* 300 (1985), 631–634.

[R4] Rosca, R. – Skew symmetric Killing vector fields on a Kenmotsu manifold, *Atti Acad. Peloritana dei Pericolanti,* 66 (1988).

[R5] Rosca, R. – ϕ-skew symmetric Killing vector fields on a Sasakian manifold, *Rend. Sem. Mat. Messina,* to appear.

[T] Tachibana, S. – On harmonic simple unit forms, *Tensor N.S.,* 27 (1973), 123–130.

[W] West, A. – *Isoparametric systems*, Geometry and Topology of Submanifolds, 222–230, World Scientific, Singapore, 1981.

[YK] Yano, K. and Kon, M. – *Anti Invariant Submanifolds*, M. Dekker, New York, 1970.

I. Mihai and L. Nicolescu,
Faculty of Mathematics,
Str. Academiei 14, 70109 Bucharest – ROMANIA

and

R. Rosca,
59 Avenue Emile Zola,
75015 Paris – FRANCE