CONVERGENCE OF APPROXIMATION PROCESSES
ON CONVEX CONES

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Abstract: The purpose of this paper is to establish convergence results for sequences of convex conic operators on $C(X; \mathcal{C})$ which are regular, i.e., sequences $\{T_n\}_{n \geq 1}$ such that for some positive linear operator $S_n$ on $C(X; \mathbb{R})$ we have $T_n(g \otimes K) = S_n(g) \otimes K$, for every continuous real valued function $g$ and every element $K$ of the convex cone $\mathcal{C}$.

1 – Introduction

We start by reviewing some of the properties of convex cones.

Definition 1. An (abstract) convex cone is a non-empty set $\mathcal{C}$ such that to every pair of elements, $K$ and $L$, of $\mathcal{C}$, there corresponds an element $K + L$, called the sum of $K$ and $L$, in such a way that addition is commutative and associative, and there exists in $\mathcal{C}$ a unique element 0, called the vertex of $\mathcal{C}$, such that $K + 0 = K$, for every $K \in \mathcal{C}$. Moreover, to every pair, $\lambda$ and $K$, where $\lambda \geq 0$ is a non-negative real number and $K \in \mathcal{C}$, there corresponds an element $\lambda K$, called the product of $\lambda$ and $K$, in such a way that multiplication is associative: $\lambda(\mu K) = (\lambda \mu) K$, $1.K = K$ and $0.K = 0$ for every $K \in \mathcal{C}$; and the distributive laws are verified: $\lambda(K + L) = \lambda K + \lambda L$, $(\lambda + \mu) K = \lambda K + \mu K$, for every $K, L \in \mathcal{C}$ and $\lambda \geq 0$, $\mu \geq 0$.

Definition 2. Let $\mathcal{C}$ be an (abstract) convex cone and let $d$ be a metric on $\mathcal{C}$. We say that the pair $(\mathcal{C}, d)$ is a metric convex cone if the following properties are valid:

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Let \((\mathcal{C}, d)\) be a metric convex cone. Then:

\[ d(\lambda K, \mu L) \leq |\lambda - \mu| d(K, 0) + \mu d(K, L), \]

for every \(K\) and \(L\) in \(\mathcal{C}\) and every \(\lambda \geq 0\) and \(\mu \geq 0\).

**Definition 3.** A non-empty subset \(\mathcal{K}\) of an (abstract) convex cone \(\mathcal{C}\) is called a **convex subcone** if \(K, L \in \mathcal{K}\) and \(\lambda \geq 0\) imply \(K + L \in \mathcal{K}\) and \(\lambda K \in \mathcal{K}\). When equipped with the induced operations, a convex subcone \(\mathcal{K} \subset \mathcal{C}\) becomes a convex cone.

**Example 1:** If \(E\) is a vector space over the reals then the set \(\mathcal{C} = \text{Conv}(E)\) of all convex non-empty subsets of \(E\) is a convex cone with the operations defined by: if \(K, L \in \text{Conv}(E)\) and \(\lambda \geq 0\)

\[ K + L = \{u + v; \ u \in K, \ v \in L\}, \]

\[ \lambda K = \{\lambda u; \ u \in K\}, \]

\[ 0 = \{\theta\}, \] where \(\theta\) is the origin of \(E\).

When \(E\) is a normed vector space, the set \(\mathcal{K}\) consisting of those elements of \(\text{Conv}(E)\) that are bounded sets is a convex subcone of \(\text{Conv}(E)\).

**Definition 4.** Let \(\mathcal{C}_1\) and \(\mathcal{C}_2\) be two convex cones. An operator \(T: \mathcal{C}_1 \to \mathcal{C}_2\) is called a **convex conic operator**, if

\[ T(F + G) = T(F) + T(G) \]

\[ T(\lambda F) = \lambda T(F) \]

for every pair \(F, G \in \mathcal{C}_1\) and every \(\lambda \geq 0\).

### 2 – Spaces of continuous functions

Let \(X\) be a compact Hausdorff space. Let \((\mathcal{C}, d)\) be a metric convex cone. We denote by \(C(X; \mathcal{C})\) the convex cone consisting of all continuous mappings \(F: X \to \mathcal{C}\). In \(C(X; \mathcal{C})\) we consider the topology of uniform convergence over \(X\),
determined by the metric defined by

\[ d(F, G) = \sup \{d(F(x), G(x)) : x \in X\} \]

for every pair \( F, G \) of elements of \( C(X; C) \). Hence \( F_n \to F \) in \( C(X; C) \) if, and only if, \( d(F_n, F) \to 0 \).

When \((C, d)\) is \( \mathbb{R} \) equipped with the usual distance \( d(x, y) = |x - y| \), then \( C(X, C) \) is the classical Banach space \( C(X) \) of all continuous real-valued functions \( f: X \to \mathbb{R} \), equipped with the sup-norm \( \|f\| = \sup\{|f(x)| : x \in X\} \).

Assume that \((X, d)\) is a metric compact space. We say that \( F: X \to C \) is a \textit{Lipschitz function} if there exists a positive constant \( M_F \) such that

\[ d(F(x), F(y)) \leq M_F \tilde{d}(x, y) \]

for all \( x, y \in X \). The subset of \( C(X; C) \) of such functions is denoted by \( \text{Lip}(X; C) \). When \((C, d)\) is \( \mathbb{R} \) equipped with usual distance \( d(x, y) = |x - y| \) we denote \( \text{Lip}(X; \mathbb{R}) = \text{Lip}(X) \) and \( \text{Lip}^+(X) = \{f \in \text{Lip}(X) : f \geq 0\} \). Notice that \( \text{Lip}(X; C) \) is a convex subcone of \( C(X; C) \).

For each \( K \in C \), we denote by \( K^* \) the element of \( C(X; C) \) defined by \( K^*(t) = K \), for all \( t \in X \).

For each \( f \in C^+(X) \) and \( K \in C \) we denote by \( f \otimes K \) the function of \( C(X; C) \) defined by \( (f \otimes K)(x) = f(x).K \), for all \( x \in X \). The convex subcone of \( C(X; C) \) generated by the functions \( f \otimes K \), where \( f \in \text{Lip}^+(X) \) and \( K \in C \), is denoted by \( \text{Lip}^+(X) \otimes C \).

**Definition 5.** Let \( K \) be a convex subcone of a convex cone \( C \). Let \( T: C(X; C) \to C(X; C) \) be a convex conic operator. We say that \( T \) is \textit{regular over} \( K \) if there exists a linear operator \( \tilde{T}: C(X; \mathbb{R}) \to C(X; \mathbb{R}) \) such that

\[ T(f \otimes K) = \tilde{T}(f) \otimes K \]

for all \( f \in C^+(X) \) and \( K \in K \).

When \( K = C \) and \( T \) is regular over \( K \), we say simply that \( T \) is \textit{regular}.

**Definition 6.** Let \( T: C(X; C) \to C(X; C) \) be a convex conic operator. We say that \( T \) is \textit{monotonically regular} if there exists a monotone linear operator \( \tilde{T}: C(X; \mathbb{R}) \to C(X; \mathbb{R}) \) such that

\[ T(f \otimes K) = \tilde{T}(f) \otimes K \]

for all \( f \in C^+(X) \) and \( K \in C \).
We recall that an operator $S$ on $C(X; \mathbb{R})$ is called monotone if $S(f) \leq S(g)$, whenever $f \leq g$. For linear operators, to be monotone is equivalent to be positive, i.e., $S(f) \geq 0$, for all $f \geq 0$.

**Remark 1.** Notice that if $T$ is regular and $\tilde{T}$ preserves the constant functions, i.e., $\tilde{T}(e_0) = e_0$, where $e_0$ denotes the real function $e_0(t) = 1$, for all $t \in X$, then $T$ also preserves the constant functions, since $T(K^*) = T(e_0 \otimes K) = \tilde{T}(e_0) \otimes K = e_0 \otimes K = K^*$, for every $K \in \mathcal{C}$.

**Definition 7.** Let $T$ be a regular operator on the convex cone $C(X; \mathcal{C})$. Define

$$\alpha(x) = \left(\tilde{T}(\tilde{d}_{x}), x\right)$$

for all $x \in X$, where $\tilde{d}_{x}$ is defined by $\tilde{d}_{x}(y) = d(x, y)$, for all $y \in X$.

**Lemma 1.** Let $(X, \tilde{d})$ be a metric compact space and $(\mathcal{C}, d)$ be a metric convex cone. Then:

**a)** If $F \in \text{Lip}^+(X) \otimes \mathcal{C}$, then $F \in \text{Lip}(X; \mathcal{C})$.

**b)** If $g \in \text{Lip}^+(X)$ and $F \in \text{Lip}^+(X) \otimes \mathcal{C}$, then the function $x \mapsto g(x)F(x)$, $x \in X$, belongs to $\text{Lip}^+(X) \otimes \mathcal{C}$.

**Proof:**

**a)** Let $F \in \text{Lip}^+(X) \otimes \mathcal{C}$ be given. There exist $g_i \in \text{Lip}^+(X)$ and $K_i \in \mathcal{C}$, for $i = 1, \ldots, m$, such that $F = \sum_{i=1}^{m} g_i \otimes K_i$. Let $M_i > 0$ be the Lipschitz constant for $g_i$, $i = 1, \ldots, m$. Then

$$d(F(x), F(y)) = d\left(\sum_{i=1}^{m} g_i(x) K_i, \sum_{i=1}^{m} g_i(y) K_i\right) \leq \sum_{i=1}^{m} d\left(g_i(x) K_i, g_i(y) K_i\right) \leq \sum_{i=1}^{m} |g_i(x) - g_i(y)| \cdot d(K_i, 0) \leq \sum_{i=1}^{m} M_i \tilde{d}(x, y) d(K_i, 0) = \left(\sum_{i=1}^{m} M_i d(K_i, 0)\right) \tilde{d}(x, y)$$

for all $x, y \in X$. Hence $F \in \text{Lip}(X; \mathcal{C})$.

**b)** Let $g \in \text{Lip}^+(X)$ and $F \in \text{Lip}^+(X) \otimes \mathcal{C}$ be given. Put $\|F\| = \sup\{d(F(x), 0); x \in X\}$. Since $F \in C(X; \mathcal{C})$ it follows that $\|F\| < \infty$. Let $M_g$ and $M_F$ be the positive constants such that

$$|g(x) - g(y)| \leq M_g \tilde{d}(x, y) \quad \text{and} \quad d(F(x), F(y)) \leq M_F \tilde{d}(x, y),$$
for all \(x, y \in X\). Then
\[
\begin{align*}
d\left(g(x) F(x), g(y) F(y)\right) & \leq |g(x) - g(y)| d(F(x), 0) + g(y) d(F(x), F(y)) \\
& \leq M_g \tilde{d}(x, y) \|F\| + \|g\| M_F \tilde{d}(x, y) \\
& = \left(\|F\| M_g + \|g\| M_F\right) \tilde{d}(x, y)
\end{align*}
\]
for all \(x, y \in X\). Hence \(gF \in \text{Lip}(X; C)\).

Now, if \(g \in \text{Lip}^+(X)\) and \(F = \sum_{i=1}^{m} g_i \otimes K_i\), where \(g_i \in \text{Lip}^+(X)\) and \(K_i \in C\), then \(gF = \sum_{i=1}^{m} h_i \otimes K_i\) where \(h_i = g \cdot g_i \in \text{Lip}^+(X)\). It follows that \(gF\) belongs to \(\text{Lip}^+(X) \otimes C\).

**Lemma 2.** Let \((X, \tilde{d})\) and \((C, d)\) be as in Lemma 1. Then \(\text{Lip}^+(X) \otimes C\) is dense in \(C(X; C)\). Consequently, \(\text{Lip}(X; C)\) is dense in \(C(X; C)\).

**Proof:** Let \(x, y \in X\), \(x \neq y\) be given. Let \(g : X \to \mathbb{R}\) be defined by \(g(z) = \tilde{d}(x, z)\), for all \(z \in X\). Since \(|g(z) - g(t)| = |\tilde{d}(x, z) - \tilde{d}(x, t)| \leq \tilde{d}(z, t)\), for all \(z, t \in X\), it follows that \(g \in \text{Lip}^+(X)\). Therefore \(h = g/\|g\|\) belongs to \(\text{Lip}(X; [0, 1])\). Moreover, \(h(y) > 0 = h(x)\), i.e., \(h\) separates \(x\) and \(y\). By Lemma 1, if \(F, G \in \text{Lip}^+(X) \otimes C\) then \(hF + (1 - h) G\) belongs to \(\text{Lip}^+(X) \otimes C\). Since \(\text{Lip}^+(X) \otimes C\) contains the constant functions, the result follows from Corollary 3, Prolla [3].

**Lemma 3** (Andrica and Mustata [1]). Let \((X, \tilde{d})\) be a metric compact space and let \(S : C(X; \mathbb{R}) \to C(X; \mathbb{R})\) be a positive linear operator. If \(f \in \text{Lip}(X)\) then there exists a positive constant \(M_f\) such that
\[
|(Sf, x) - f(x) (Se_0, x)| \leq M_f \alpha(x)
\]
for all \(x \in X\).

**Proof:** Let \(f \in \text{Lip}(X)\) and let \(M_f > 0\) be a Lipschitz constant for \(f\), i.e.,
\[
|f(x) - f(y)| \leq M_f \tilde{d}(x, y)
\]
for all \(x, y \in X\). It follows that
\[
-M_f \tilde{d}(x, \cdot) \leq f(\cdot) - f(x) e_0 \leq M_f \tilde{d}(x, \cdot)
\]
for all \(x \in X\). Since \(S\) is linear and positive we have
\[
-M_f (S(\tilde{d}_x), x) \leq (Sf, x) - f(x) (Se_0, x) \leq M_f (S(\tilde{d}_x), x)
\]
for all $x \in X$. Therefore

$$\left| (Sf, x) - f(x) (Se_0, x) \right| \leq M_f (S(\tilde{d}_x), x)$$

for all $x \in X$. \hfill \blacksquare

**Corollary 1.** Let $(X, \tilde{d})$ and $S$ be as in Lemma 3. Assume that $Se_0 = e_0$. If $f \in \text{Lip}(X)$ then there exists a positive constant $M_f$ such that

$$\left| (Sf, x) - f(x) \right| \leq M_f \alpha(x)$$

for all $x \in X$.

**Proof:** It follows immediately from Lemma 3 since $(Se_0, x) = 1$, for all $x \in X$. \hfill \blacksquare

**Remark 2.** A positive linear operator $S$ on $C(X; \mathbb{R})$ such that $S\alpha = \alpha_0$, i.e., $S$ preserves the constant functions, is called a Markov operator on $C(X; \mathbb{R})$. Andrica and Mustata [1] proved Lemma 3 assuming that $S$ is a Markov operator.

**Proposition 1.** Let $(X, \tilde{d})$ be a metric compact space and $(C, d)$ be a metric convex cone. Let $T$ be a monotonically regular operator on $C(X; C)$ and let $F \in \text{Lip}^+(X) \otimes C$ be given. There exist positive constants $M_F$ and $A_F$ such that

$$d(TF, x), F(x) \leq M_F \alpha(x) + A_F |(\tilde{T}e_0, x) - 1|$$

for all $x \in X$.

**Proof:** Let $F = \sum_{i=1}^m g_i \otimes K_i$ be given, where $g_i \in \text{Lip}^+(X)$ and $K_i \in C$, for $i = 1, \ldots, m$. Since $T$ is convex conic and regular, we have

$$(TF, x) = \left( \sum_{i=1}^m T(g_i \otimes K_i), x \right) = \sum_{i=1}^m (\tilde{T}(g_i), x) K_i$$

for all $x \in X$.

For each $i = 1, \ldots, m$, by Lemma 3, there exists a constant $M_i > 0$ such that

$$\left| (\tilde{T}(g_i), x) - g_i(x) (\tilde{T}e_0, x) \right| \leq M_i \alpha(x)$$

for all $x \in X$. Let $M_F$ and $A_F$ be the positive constants defined by

$$M_F = \sum_{i=1}^m M_i d(K_i, 0) \quad \text{and} \quad A_F = \sum_{i=1}^m \|g_i\| d(K_i, 0).$$
Then,
\[ d\left((TF, x), F(x)\right) \leq \sum_{i=1}^{m} d\left((\hat{T}(g_i), x)K_i, g_i(x)K_i\right) \leq \sum_{i=1}^{m} |(\hat{T}(g_i), x) - g_i(x)| d(K_i, 0) \leq \sum_{i=1}^{m} \left[ M_i \alpha(x) + \|g_i\| \cdot |(\hat{T}\epsilon_0, x) - 1| \right] d(K_i, 0) \leq M_F \alpha(x) + A_F |(\hat{T}\epsilon_0, x) - 1| \]
for all \( x \in X \).

**Corollary 2.** Let \((X, \tilde{d}), (\mathcal{C}, d)\) and \(T\) be as in Proposition 1. Assume that \(\hat{T}\) preserves the constant functions. If \(F \in \text{Lip}^+(x) \otimes \mathcal{C}\) then there exists a positive constant \(M_F\) such that
\[ d\left((TF, x), F(x)\right) \leq M_F \alpha(x) \]
for all \( x \in X \).

**Proof:** The result follows from Proposition 1 since \(\hat{T}(\epsilon_0) = \epsilon_0\).

**Definition 8.** Let \(\{T_n\}_{n \geq 1}\) be a sequence of operators on \(C(X; \mathcal{C})\). We say that \(\{T_n\}_{n \geq 1}\) is **uniformly equicontinuous** if for each \(\varepsilon > 0\) there exists \(\delta > 0\) such that \(d(F, G) < \delta\) implies \(d(T_n F, T_n G) < \varepsilon\), for all \(n = 1, 2, 3, \ldots\).

Let \(\{T_n\}_{n \geq 1}\) be a sequence of regular operators on \(C(X; \mathcal{C})\). For each \(n \geq 1\) we denote by \(\alpha_n\) the function defined by
\[ \alpha_n(x) = (\hat{T}_n(\tilde{d}_x), x) \]
for all \(x \in X\).

**Theorem 1.** Let \((X, \tilde{d})\) be a metric compact space and \((\mathcal{C}, d)\) be a metric convex cone. Let \(\{T_n\}_{n \geq 1}\) be a sequence of monotonically regular operators on \(C(X; \mathcal{C})\). Assume that \(\{T_n\}_{n \geq 1}\) is uniformly equicontinuous. If \(\hat{T}_n \epsilon_0 \to \epsilon_0\) and \(\{\alpha_n(x)\}_{n \geq 1}\) converges to zero, uniformly in \(x \in X\), then \(T_n F \to F\), for every \(F \in \text{Lip}^+(X) \otimes \mathcal{C}\).

**Proof:** Let \(G \in \text{Lip}^+(X) \otimes \mathcal{C}\) be given. By Proposition 1, there exist positive constants \(M_G\) and \(A_G\) such that, for each \(n \geq 1\),
\[ d\left((T_n G, x), G(x)\right) \leq M_G \alpha_n(x) + A_G |(\hat{T}_n \epsilon_0, x) - 1| \]
for all \(x \in X\). Since \(\alpha_n(x) \to 0\), uniformly in \(x \in X\) and \(\hat{T}_n \epsilon_0 \to \epsilon_0\) it follows that \(d(T_n G, G) \to 0\). Hence \(T_n G \to G\), for each \(G\) in \(\text{Lip}^+(X) \otimes \mathcal{C}\).
Let $F \in C(X;C)$ and $\varepsilon > 0$ be given. By the uniform equicontinuity of the sequence $\{T_n\}_{n \geq 1}$, there is some $\delta > 0$, which we may assume to verify $\delta < \varepsilon/3$, such that $d(F, H) < \delta$ implies $d(T_nF, T_nH) < \varepsilon/3$, for all $n \geq 1$. By Lemma 2, there exists $G$ in $\text{Lip}^+(X) \otimes C$ such that $d(F, G) < \delta$. Since $T_nG \to G$ as proved above, there is $n_0$ such that $n \geq n_0$ implies $d(T_nG, G) < \varepsilon/3$. It follows that, for $n \geq n_0$

$$
\begin{align*}
\left| d(\left( T_n F, x \right), F(x) ) - d \left( \left( T_n G, x \right), G(x) \right) \right| & \leq \left( d(T_n F, T_n G) + d(T_n G, G) + d(G, F) \right) < \varepsilon \\
& \leq \frac{\varepsilon}{3} \quad \text{for all } x \in X.
\end{align*}
$$

for all $x \in X$. Hence $T_n F \to F$.

**Remark 3.** If each $\widehat{T}_n$ preserves the constant functions, then the proof of Theorem 1 implies that

$$
d(T_n F, F) \leq M_F \|\alpha_n\|
$$

for all $n \geq 1$ and all $F \in \text{Lip}^+(X) \otimes C$, where $\|\alpha_n\| = \sup\{|\alpha_n(x)|; x \in X\}$.

If we define $\beta_n(x) = (\widehat{T}_n(\widehat{\partial}_{x}^2) x)$, for all $x \in X$, then we have that $\|\alpha_n\| \leq \frac{\|\beta_n\|}{\varepsilon}$, for all $n \in \mathbb{N}$, and the following result holds:

**Corollary 3.** Let $\{T_n\}_{n \geq 1}$ be as in Theorem 1. Assume that each $\widehat{T}_n$ preserves the constant functions. If $\{\beta_n(x)\}_{n \geq 1}$ converges to zero, uniformly in $x \in X$, then $T_n F \to F$, for every $F \in C(X;C)$. Furthermore, if $F \in \text{Lip}^+(X) \otimes C$ then there exists a constant $M_F > 0$ such that

$$
d(T_n F, F) \leq M_F \|\beta_n\|^{\frac{1}{2}}
$$

for all $n \geq 1$.

**Proof:** Apply Theorem 1 and Remark 3.

**Example 2:** Let $J$ be a finite set, and for each $k \in J$, let $t_k \in X$ and $\psi_k \in C^+(X)$ be given. The convex conic operator $T$ defined on $C(X;C)$ by

$$
(TF, x) = \sum_{k \in J} \psi_k(x) F(t_k)
$$

for all $F \in C(X;C)$ and $x \in X$ is called an operator of interpolation type. If $F = f \otimes K$, where $f \in C^+(X)$ and $K \in C$, then

$$
(TF, x) = \sum_{k \in J} \psi_k(x) [f(t_k)K] = \left[ \sum_{k \in J} \psi_k(x) f(t_k) \right] K.
$$
Hence, $T$ is regular and $T(f \otimes K) = \hat{T}(f) \otimes K$ where, for each $f \in C(X; \mathbb{R})$,

$$(\hat{T}f, x) = \sum_{k \in J} \psi_k(x) f(t_k).$$

Let us assume that, for every $x \in X$,

$$\sum_{k \in J} \psi_k(x) = 1.$$

It follows that $\hat{T}e_0 = e_0$. The operators of Bernstein and of Hermite–Fejér type are examples of operators satisfying such condition.

**Remark 4.** If $(C, d)$ is a convex cone and $T$ is a regular operator on $C(X; C)$ then $TK^* = T(e_0 \otimes K) = \hat{T}(e_0) \otimes K$, for every $K \in C$, and we have

$$d\left((TK^*, x), K^*(x)\right) = d\left((\hat{T}e_0, x)K, e_0(x)K\right) \leq |(\hat{T}e_0, x) - 1| d(K, 0)$$

for all $x \in X$. It follows that if $\{T_n\}_{n \geq 1}$ is a sequence of regular operators on $C(X; C)$ such that $\hat{T}_n e_0 \to e_0$, then $T_n K^* \to K^*$, for every $K \in C$.

**Lemma 4.** Let $(X, \hat{d})$ be a metric compact space and $(C, d)$ be a convex cone. Let $\{T_n\}_{n \geq 1}$ be a sequence of regular convex conic operator on $C(X; C)$. Assume that $\hat{T}_n e_0 \to e_0$. If $F \in C(X; C)$ then $(T_n[F(x)]^*, x) \to F(x)$, uniformly in $x \in X$.

**Proof:** Let $F \in C(X; C)$ and $\varepsilon > 0$ be given. Since $\hat{T}_n e_0 \to e_0$ there is $n_0$ such that $n \geq n_0$ implies

$$\left|(\hat{T}_n(e_0), x) - 1\right| < \frac{\varepsilon}{2 \|F\|}$$

for all $x \in X$, where $\|F\| = \sup\{d(F(x), 0); x \in X\}$. It follows that, for $n \geq n_0$

$$d\left((T_n[F(x)]^*, x), f(x)\right) \leq \left|(\hat{T}_n(e_0), x) - 1\right| d(F(x), 0) \leq \left(\frac{\varepsilon}{2 \|F\|}\right) \cdot \|F\| < \varepsilon$$

for all $x \in X$. Therefore, $(T_n[F(x)]^*, x) \to F(x)$, uniformly in $x \in X$. \qed
3 – Hausdorff convex cones

Definition 9. An ordered convex cone is a pair \((C, \preceq)\), where \(C\) is an (abstract) convex cone and \(\preceq\) is an ordering of its elements, i.e., \(\preceq\) is a reflexive, transitive and antisymmetric relation on \(C\), in such a way that

a) \(K \leq L\) implies \(K + M \leq L + M\), for every \(M \in C\),

b) \(K \leq L, \lambda \geq 0\) implies \(\lambda K \leq \lambda L\),

c) \(\lambda \leq \mu\) implies \(\lambda K \leq \mu K\), for every \(K \geq 0\).

Definition 10. Let \((C, \preceq)\) be an ordered convex cone and let \(d_H\) be a semi-metric on \(C\). We say that \(d_H\) is a Hausdorff semi-metric on \(C\) if there exists an element \(B \geq 0\) on \(C\) such that:

a) For every pair \(K, L \in C\) and \(\lambda \geq 0\), the following is true: \(d_H(K, L) \leq \lambda\) if, and only if, \(K \leq L + \lambda B\) and \(L \leq K + \lambda B\),

b) \(\lambda B \leq \mu B\) implies \(\lambda \leq \mu\).

If \(d_H\) is a Hausdorff semi-metric on \(C\), we say that \((C, d_H)\) is a Hausdorff convex cone.

Example 3: If \(C = \mathbb{R}\) with the usual operations and ordering, the usual distance \(d_H(x, y) = |x - y|\) is a Hausdorff metric on \(\mathbb{R}\), with \(B = 1\).

Example 4: Let \(C\) be the convex subcone of \(\text{Conv}(E)\) of all elements of \(\text{Conv}(E)\) that are bounded sets and let \(B\) be the closed unit ball of \(E\). Define on \(C\) the usual Hausdorff semi-metric \(d_H\) by setting

\[
d_H(K, L) = \inf \left\{ \lambda > 0; \ K \subset L + \lambda B, \ L \subset K + \lambda B \right\}
\]

for every pair \(K, L \in C\). Then \((C, d_H)\) is a Hausdorff convex cone.

Let \((X, \tilde{d})\) be a metric compact space and \((C, d_H)\) be a Hausdorff convex cone. In \(C(X; C)\) we consider the topology determined by the metric defined by

\[
d(F, G) = \sup \left\{ d_H(F(x), G(x)); \ x \in X \right\}
\]

for every pair \(F, G\) in \(C(X; C)\).

Remark 5. If \((C, d_H)\) is a Hausdorff convex cone and \(\{T_n\}_{n \geq 1}\) is a sequence of regular operators on \(C(X; C)\) then \(T_n B^* \rightarrow B^*\) implies \(T_n e_0 \rightarrow e_0\). Indeed, let
\( \varepsilon > 0 \) be given. Since \( B^* = e_0 \otimes B \) and \( T_n(e_0 \otimes B) \to e_0 \otimes B \), it follows that there is \( n_0 \) such that \( n \geq n_0 \) implies
\[
d_H\left( (\tilde{T}_n(e_0), x) B, e_0(x) B \right) < \varepsilon
\]
for all \( x \in X \). By the definition of \( d_H \) we have
\[
\left( \tilde{T}_n(e_0), x \right) B \leq B + \varepsilon B = (1 + \varepsilon) B
\]
and
\[
B \leq (\tilde{T}_n(e_0), x) B + \varepsilon B
\]
for all \( x \in X \). By condition b) of Definition (10) we have \( (\tilde{T}_n(e_0), x) < 1 + \varepsilon \) and \( 1 - \varepsilon < (\tilde{T}_n(e_0), x) \), for all \( x \in X \). Hence \( |(\tilde{T}_n(e_0), x) - 1| < \varepsilon \), for all \( x \in X \) and so \( \tilde{T}_n e_0 \to e_0 \).

We recall that an operator \( T \) on \( C(X; \mathbb{C}) \) is called \emph{monotone}, if \( F \leq G \) implies \( TF \leq TG \) for every pair \( F, G \) in \( C(X; \mathbb{C}) \).

**Remark 6.** If \( (C, d_H) \) is a Hausdorff convex cone and \( T \) is a regular operator on \( C(X; \mathbb{C}) \) that is monotone then \( \tilde{T} \) is also monotone. Indeed, for \( f, g \in C(X; \mathbb{R}) \) such that \( f \leq g \) we have \( f \otimes B \leq g \otimes B \). It follows that \( T(f \otimes B) \leq T(g \otimes B) \), and since \( T \) is regular, we get \( (\tilde{T}(f), x)B \leq (\tilde{T}(g), x)B \), for all \( x \in X \). Therefore \( \tilde{T}f \leq \tilde{T}g \).

**Theorem 2.** Let \( (X, \tilde{d}) \) be a metric compact space and \( (C, d_H) \) be a Hausdorff convex cone. Let \( \{T_n\}_{n \geq 1} \) be a sequence of regular continuous operators on \( C(X; \mathbb{C}) \). Assume that each \( T_n \) is monotone and \( T_n B^* \to B^* \). If \( \{a_n(x)\}_{n \geq 1} \) converges to zero, uniformly in \( x \in X \), then \( T_n F \to F \), for every \( F \in C(X; \mathbb{C}) \).

**Proof:** By Theorem 1 it suffices to show that the sequence \( \{T_n\}_{n \geq 1} \) is uniformly equicontinuous. Let \( \varepsilon > 0 \) be given. Choose \( \delta_0 > 0 \) such that \( \delta_0(1 + \delta_0) < \varepsilon \). Since \( T_n B^* \to B^* \), there is \( n_0 \) so that \( n > n_0 \) implies \( d_H((T_n B, x), B) < \delta_0 \), for all \( x \in X \). It follows from the definition of \( d_H \) that
\[
(T_n B^*, x) \leq B + \delta_0 B = (1 + \delta_0) B
\]
and
\[
B \leq (T_n B^*, x) + \delta_0 B
\]
for all \( x \in X \), and \( n > n_0 \).
Let $F, G \in C(X; \mathcal{C})$ be such that $d(F, G) < \delta_0$. We claim that $d(T_n F, T_n G) < \varepsilon$, for all $n > n_0$. Indeed, since $d_H(F(x), G(x)) < \delta_0$, for all $x \in X$, it follows that $F \leq G + \delta_0 B^*$ and $G \leq F + \delta_0 B^*$.

Since each $T_n$ is convex conic and monotone, we have, for each $n \geq 1$, $T_n F \leq T_n G + \delta_0 T_n B^*$ and $T_n G \leq T_n F + \delta_0 T_n B^*$. Therefore, for each $n \geq 1$, $(T_n F, x) \leq (T_n G, x) + \delta_0 (T_n B^*, x)$, for all $x \in X$. It follows that, for $n > n_0$

$$
(T_n F, x) \leq (T_n G, x) + \delta_0 (1 + \delta_0) B < (T_n G, x) + \varepsilon B
$$

for all $x \in X$. Similarly, for $n > n_0$

$$
(T_n G, x) < (T_n F, x) + \varepsilon B
$$

for all $x \in X$. Hence, for all $n > n_0$

$$
d_H((T_n F, x), (T_n G, x)) < \varepsilon
$$

for all $x \in X$. It follows that, for all $n > n_0$

$$
d(T_n F, T_n G) < \varepsilon.
$$

On the other hand, since each $T_n$ is continuous, there exist $\delta_1, \ldots, \delta_{n_0}$ such that $d(F, G) < \delta_k$ implies $d(T_k F, T_k G) < \varepsilon$, for $k = 1, 2, \ldots, n_0$. Let $\delta = \min\{\delta_0, \delta_1, \ldots, \delta_{n_0}\}$. Clearly $d(F, G) < \delta$ implies $d(T_n F, T_n G) < \varepsilon$, for all $n = 1, 2, 3, \ldots$.

**Corollary 4.** Let $(X, \tilde{d})$, $(\mathcal{C}, d_H)$ and $\{T_n\}_{n \geq 1}$ be as in Theorem 2. Assume that $T_n$ preserves the constant functions. If $\{\beta_n(x)\}_{n \geq 1}$ converges to zero, uniformly in $x \in X$, then $T_n F \rightharpoonup F$, for every $F \in C(X; \mathcal{C})$. Furthermore, if $F \in \text{Lip}^+(X) \otimes \mathcal{C}$ then there exists a constant $M_F > 0$ such that

$$
d(T_n F, F) \leq M_F \|\beta_n\|^{\frac{1}{2}}
$$

for all $n = 1, 2, 3, \ldots$, where $\beta_n(x) = (\tilde{T}_n(\tilde{d}_x)^2, x)$, for all $x \in X$.

Let us recall that the modulus of continuity of $F \in C(X; \mathcal{C})$ is defined as

$$
w(F, \delta) = \sup\{d(F(x), F(t)); x, t \in X, \tilde{d}(x, t) \leq \delta\}
$$

for every $\delta > 0$. By uniform continuity of $F$, we have $w(F, \delta) \to 0$ as $\delta \to 0$.

Let us consider the following condition:
There exists a constant $p$ with $0 < p \leq 1$ such that $w(F, \lambda \delta) \leq [1 + \lambda^\frac{1}{p}] w(F, \delta)$, for all $F \in C(X; C)$ and all $\delta, \lambda > 0$.

If $X$ is a compact convex subset of a $q$-normed linear space with $0 < q \leq 1$, then (*) holds for $p = q$.

The following result is proved in [4]:

**Lemma 5.** Assume that (*) holds. Let $F \in C(X; C)$ and $\delta > 0$ be given. Then

$$d_H(F(x), F(t)) \leq \left[1 + \left(\frac{\tilde{d}(x,t)}{\delta}\right)^{\frac{1}{p}}\right] w(F, \delta)$$

for every pair, $x$ and $t$, of elements of $X$.

If $\{T_n\}_{n \geq 1}$ is a sequence of convex conic operators on $C(X; C)$ that are regular, let

$$a_n(x) = (\tilde{T}_n((\tilde{d}_x)^{\frac{1}{p}}), x)$$

for all $x \in X$, where $p$ is given by condition (*).

**Proposition 2.** Assume that (*) holds. Let $\{T_n\}_{n \geq 1}$ be a sequence of convex conic operators on $C(X; C)$ such that each $T_n$ is monotone and regular. Then

$$d_H\left((T_n F, x), F(x)\right) \leq \left[\tilde{T}_n(e_0), x\right] + \frac{1}{\delta^\frac{1}{p}} a_n(x) w(F, \delta) + d_H\left((T_n[F(x)]^*, x), F(x)\right)$$

for every $F \in C(X; C)$, $x \in X$ and $\delta > 0$.

**Proof:** Let $F \in C(X; C)$ and $\delta > 0$ be given. By Lemma 5, for $t, x \in X$

$$F(t) \leq F(x) + \left[1 + \left(\frac{\tilde{d}(x,t)}{\delta}\right)^{\frac{1}{p}}\right] w(F, \delta) B$$

$$= F(x) + w(F, \delta) \left[B + \frac{1}{\delta^\frac{1}{p}} (\tilde{d}(x,t))^{\frac{1}{p}} \otimes B\right]$$

Hence,

$$F \leq [F(x)]^* + w(F, \delta) \left[B^* + \frac{1}{\delta^\frac{1}{p}} (\tilde{d}_x)^{\frac{1}{p}} \otimes B\right].$$

Since each $T_n$ is monotone and regular we have

$$(T_n F, x) \leq (T_n[F(x)]^*, x) + w(F, \delta) \left[\tilde{T}_n(e_0), x\right] + \frac{1}{\delta^\frac{1}{p}} a_n(x) B$$
for all \( x \in X \). Similarly,
\[
\left( T_n[F(x)]^*, x \right) \leq \left( T_n F, x \right) + w(F, \delta) \left[ (\hat{T}_n(e_0), x) + \frac{1}{\delta^p} a_n(x) \right] B
\]
for all \( x \in X \). Therefore
\[
d_H \left( (T_n F, x), (T_n[F(x)]^*, x) \right) \leq w(F, \delta) \left[ (\hat{T}_n(e_0), x) + \frac{1}{\delta^p} a_n(x) \right].
\]
for all \( x \in X \).

**Theorem 3.** Let \((X, \tilde{d})\) be a compact metric space and \((\mathcal{C}, d_H)\) be a Hausdorff convex cone. Let \( \{T_n\}_{n \geq 1} \) be a sequence of convex conic operators on \( C(X; \mathcal{C}) \) such that each \( T_n \) is monotone and regular. Assume that (*) holds and that
\begin{itemize}
  \item[i)] \( T_n B^* \to B^* \),
  \item[ii)] \( a_n(x) = 0(\frac{1}{n}) \), uniformly \( x \in X \).
\end{itemize}
Then \( T_n F \to F \), for every \( F \in C(X; \mathcal{C}) \).

**Proof:** Let \( F \in C(X; \mathcal{C}) \) and \( \varepsilon > 0 \) be given. By i), Remark 5 and Lemma 4 choose \( n_1 \) so that \( n \geq n_1 \) implies
\begin{itemize}
  \item[(1)] \( (\hat{T}_n(e_0), x) < 1 + \varepsilon/2 \),
  \item[(2)] \( d_H ((T_n[F(x)]^*, x), F(x)) < \varepsilon/2 \),
\end{itemize}
for all \( x \in X \). By ii) there is some constant \( k > 0 \) such that
\begin{itemize}
  \item[(3)] \( n a_n(x) \leq k \),
\end{itemize}
for \( n = 1, 2, \ldots \) and all \( x \in X \). Since \( w(F, \delta) \to 0 \) as \( \delta \to 0 \), we can choose \( n_2 \) such that \( n \geq n_2 \) implies
\begin{itemize}
  \item[(4)] \( w(F, n^{-p}) < (\varepsilon/2) (1 + k + \varepsilon/2)^{-1} \).
\end{itemize}
By Proposition 2 and (1)--(4), it follows that for \( n \geq n_0 = \max\{n_1, n_2\} \)
\[
d_H \left( (T_n F, x), F(x) \right) \leq \left[ (\hat{T}_n(e_0), x) + \frac{1}{\delta^p} a_n(x) \right] w(F, \delta) + d_H \left( (T_n[F(x)]^*, x), F(x) \right)
\leq \left[ (\hat{T}_n(e_0), x) + n a_n(x) \right] w(F, n^{-p}) + d_H \left( (T_n[F(x)]^*, x), F(x) \right)
< (1 + k + \varepsilon/2) w(F, n^{-p}) + \varepsilon/2 < \varepsilon
\]
for all \( x \in X \). Hence \( T_n F \to F \). ■

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