NOTE ON THE CHEBYSHEV POLYNOMIALS
AND APPLICATIONS TO THE FIBONACCI NUMBERS

José Morgado

Abstract: In [12], Gheorghe Udrea generalizes a result obtained in [8], by showing that, if \((U_n)_{n \geq 0}\) is the sequence of Chebyshev polynomials of the second kind, then the product of any two distinct elements of the set
\[
\left\{ U_n, U_{n+2r}, U_{n+4r}, 4U_{n+r}U_{n+2r}U_{n+3r} \right\}, \quad r, n \in \mathbb{N},
\]
increased by \(U_a^2U_b^2\), for suitable nonnegative integers \(a\) and \(b\), is a perfect square.

In this note, one obtains a similar result for the Chebyshev polynomials of the first kind and one states some generalizations of results contained in [12] and in [8].

1 – Preliminaries

Diophantus raised the following problem ([4], pp. 179–181):

“To find four numbers such that the product of any two increased by unity is a square”,

for which he obtained the solution \(\frac{1}{16}, \frac{33}{16}, \frac{68}{16}, \frac{105}{16}\).

Fermat ([3], p. 251) found the solution 1, 3, 8, 120.

In 1968, J.H. van Lint raised the problem whether the number 120 is the unique (positive) integer \(n\) for which the set \{1, 3, 8, 120\} constitutes a solution for Diophantus’ problem above; in the same year, A. Baker and H. Davenport [1] studied this question and concluded that, in fact, 120 is the unique integer satisfying the problem raised by J.H. van Lint.

In 1977, V.E. Hoggatt and G.E. Bergum [5] observed that 1, 3, 8 are, respectively, the terms \(F_2, F_4, F_6\), of the Fibonacci sequence \((F_n)_{n \geq 0}\), defined by the conditions
\[
F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n, \quad n \geq 0,
\]

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and formulated the problem of finding a positive integer $n$ such that

$$
F_{2t}n + 1, \quad F_{2t+2}n + 1, \quad F_{2t+4}n + 1
$$

be perfect squares.

Hoggatt and Bergum obtained the number

$$
n = 4 F_{2t+1} F_{2t+2} F_{2t+3},
$$

which, for $t = 1$, gives exactly $n = 120$.

In 1984, this result was generalized ([8], p. 443), by showing that the product of any two distinct elements of the set

$$
\left\{ F_n, F_{n+2r}, F_{n+4r}, 4F_{n+r}F_{n+2r}F_{n+3r} \right\},
$$

increased by $\pm F_a^2 F_b^2$ (for suitable integers $a$ and $b$) is a perfect square, i.e., this set is a Fibonacci quadruple.

In 1987, this result was generalized by A.F. Horadam [6], who proved that the product of any two distinct elements of the set

$$
\left\{ w_n, w_{n+2r}, w_{n+4r}, 4w_{n+r}w_{n+2r}w_{n+3r} \right\},
$$

increased by a suitable integer, is a perfect square, i.e., this set is a Diophantine quadruple, not being necessarily a Fibonacci quadruple.

The sequence $(w_n)_{n \geq 0}$ was introduced, in 1965, by A.F. Horadam [7]:

$$
w_n = w_n(a, b; p, q), \quad w_0 = a, \quad w_1 = b \quad \text{and} \quad w_n = p w_{n-1} - q w_{n-2},
$$

with $a$, $b$, $p$, $q$ integers, and $n \geq 2$. This sequence generalizes the sequence $(F_n)_{n \geq 0}$, since one has $F_n = w_n(0, 1; 1, -1)$.

In the paper of Gheorghe Udrea [12], one obtains another generalization of the result contained in [8], by means of the Chebyshev polynomials of the second kind.

The sequence of Chebyshev polynomials of the first kind is the sequence $(T_n(x))_{n \geq 0}$, where $x \in \mathbb{C}$, defined by the recurrence relation

$$
(1.1) \quad T_{n+2}(x) = 2x T_{n+1} - T_n(x),
$$

with $T_0(x) = 1$ and $T_1(x) = x$. Thus, one has

$$
T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \quad T_4(x) = 8x^4 - 8x^2 + 1, \quad \ldots.
$$
The sequence of Chebyshev polynomials of the second kind is the sequence 
\((U_n(x))_{n \geq 0}\), where \(x \in \mathbb{C}\), defined by the same recurrence relation 
\[ U_{n+2}(x) = 2x U_{n+1}(x) - U_n(x), \]
with \(U_0(x) = 1\), and \(U_1(x) = 2x\). Thus, one has 
\[ U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x, \quad U_4(x) = 16x^4 - 12x^2 + 1, \quad \ldots. \]

The (ordinary) generating function of \( (T_n(x))_{n \geq 0} \) is the formal series 
\[ g_1(y) = T_0(x) + T_1(x)y + T_2(x)y^2 + \ldots + T_n(x)y^n + \ldots. \]

By taking into account the recurrence relation (1.1), we are led to consider the reducing polynomial 
\[ k(y) = 1 - 2xy + y^2. \]

One has clearly 
\[ g_1(y) k(y) = \left[ T_0(x) + T_1(x)y + T_2(x)y^2 + \ldots + T_n(x)y^n + \ldots \right] (1 - 2xy + y^2) \]
\[ = T_0(x) + \left[ T_1(x) - 2xT_0(x) \right] y + \ldots \]
\[ + \ldots + \left[ T_n(x) - 2xT_{n-1}(x) + T_{n-2}(x) \right] y^n + \ldots = 1 - xy, \]
since, by (1.1), \( T_n(x) - 2xT_{n-1}(x) + T_{n-2}(x) \) is the zero polynomial for \( n \geq 2 \).

Thus, one obtains the generating function, \( g_1(y) \), under a finite form, 
\[ g_1(y) = \frac{1 - xy}{1 - 2xy + y^2}, \]
which can be written as 
\[ g_1(y) = \frac{1 - xy}{\left[ y - (x + \sqrt{x^2 - 1}) \right] \left[ y - (x - \sqrt{x^2 - 1}) \right]} = \frac{A}{y - (x + \sqrt{x^2 - 1})} + \frac{B}{y - (x - \sqrt{x^2 - 1})}, \]
where 
\[ \begin{cases} 
A + B = -x, \\
A(x - \sqrt{x^2 - 1}) + B(x + \sqrt{x^2 - 1}) = -1.
\end{cases} \]

From this, it follows (with \( x \neq \pm 1 \)) that 
\[ A = \frac{1 - x^2 - x\sqrt{x^2 - 1}}{2\sqrt{x^2 - 1}}, \quad \text{and} \quad B = \frac{1 - x^2 + x\sqrt{x^2 - 1}}{2\sqrt{x^2 - 1}}, \]
and, consequently,

\[
g_1(y) = \frac{1}{2\sqrt{x^2-1}} \left[ \frac{\sqrt{x^2-1}}{1 - (x + \sqrt{x^2-1}) y} + \frac{\sqrt{x^2-1}}{1 - (x - \sqrt{x^2-1}) y} \right]
\]

\[
= \frac{1}{2} \left[ 1 + (x + \sqrt{x^2-1}) y + (x + \sqrt{x^2-1})^2 y^2 + \ldots + (x + \sqrt{x^2-1})^n y^n + \ldots \right]
\]

\[
+ \frac{1}{2} \left[ 1 + (x - \sqrt{x^2-1}) y + (x - \sqrt{x^2-1})^2 y^2 + \ldots + (x - \sqrt{x^2-1})^n y^n + \ldots \right] .
\]

Since, by (1.3) \( T_n(x) \) is the coefficient of \( y^n \), one concludes that

\[
T_n(x) = \frac{1}{2} \left[ (x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n \right] .
\]

For the Chebyshev polynomials of the second kind, one finds, by a similar way, the corresponding generating function, under a finite form (with \( x \neq \pm 1 \)):

\[
g_2(y) = \frac{1}{y^2 - 2xy + 1}
\]

\[
= \frac{1}{2\sqrt{x^2-1}} \left[ \frac{x + \sqrt{x^2-1}}{1 - (x + \sqrt{x^2-1}) y} - \frac{x - \sqrt{x^2-1}}{1 - (x - \sqrt{x^2-1}) y} \right]
\]

and one obtains, after the developments in power series of

\[
\frac{x + \sqrt{x^2-1}}{1 - (x + \sqrt{x^2-1}) y} \quad \text{and} \quad \frac{x - \sqrt{x^2-1}}{1 - (x - \sqrt{x^2-1}) y},
\]

\[
U_n(x) = \frac{1}{2\sqrt{x^2-1}} \left[ (x + \sqrt{x^2-1})_{n+1} - (x - \sqrt{x^2-1})_{n+1} \right] .
\]

Since, for each \( x \in \mathbb{C} \), there is some \( \theta \in \mathbb{C} \) such that \( x = \cos \theta \), one can write

\[
T_n(\cos \theta) = \frac{1}{2} \left[ (\cos \theta + i \sin \theta)^n + (\cos \theta - i \sin \theta)^n \right] = \cos n \theta ,
\]

\[
U_n(\cos \theta) = \frac{1}{2i \sin \theta} \left[ (\cos \theta + i \sin \theta)^{n+1} - (\cos \theta - i \sin \theta)^{n+1} \right] = \frac{\sin(n+1) \theta}{\sin \theta} .
\]

By means of the relations (1.6) and (1.7), it is easy to see that the following connections, between the two kinds of Chebyshev polynomials, hold:

\[
T_n(x) = U_n(x) - xU_{n-1}(x), \quad n \geq 1 ,
\]

\[
(1 - x^2) U_n(x) = x T_{n+1}(x) - T_{n+2}(x), \quad n \geq 0 ,
\]

\[
T_{n+1}(x) = 1 + (x^2 - 1) U_n^2(x), \quad n \geq 0 .
\]
The Chebyshev polynomials, \( T_n(x) \) and \( U_n(x) \), are special ultraspherical (or Gegenbauer) polynomials. The ultraspherical polynomials are special cases of the Jacobi polynomials, i.e., of the polynomials \( P_n^{(\alpha,\beta)}(x) \) such that ([11], pp. 71–73),

\[
P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n (1-x)^{-\alpha} (1+x)^{-\beta}}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} \left[ (1-x)^{\alpha+n} \cdot (1+x)^{\beta+n} \right].
\]

The ultraspherical polynomials are the Jacobi polynomials, for which one has \( \alpha = \beta \); for the Chebyshev polynomials of the first kind, one has \( \alpha = \beta = -\frac{1}{2} \) and, for the Chebyshev polynomials of the second kind, one has \( \alpha = \beta = \frac{1}{2} \).

By taking into account (1.6), it is natural to extend the meaning of \( T_n \) for \( n < 0 \): one puts

\[
T_{-r}(x) = T_{-r}(\cos \theta) = \cos(-r) \theta = \cos r \theta = T_r(\cos \theta) = T_r(x).
\]

2 – Some properties of the Chebyshev polynomials of the first kind

In order to obtain, for the Chebyshev polynomials of the first kind, a result analogous to that obtained by Gheorghe Udrea for the Chebyshev polynomials of the second kind, we need to prove the following lemma:

**Lemma 1.** If \( (T_n(x))_{n \geq 0} \) is the sequence of Chebyshev polynomials of the first kind, then one has:

\[
(2.1) \quad T_n(x) T_{n+r+s}(x) + \frac{1}{2} \left[ T_{r-s}(x) - T_{r+s}(x) \right] = T_{n+r}(x) T_{n+s}(x).
\]

\[
(2.2) \quad 4T_n(x) T_{n+r}(x) T_{n+s}(x) T_{n+r+s}(x) + \frac{1}{4} \left[ T_{r-s}(x) - T_{r+s}(x) \right]^2 =
\]

\[
= \left[ T_n(x) T_{n+r+s}(x) + T_{n+r} T_{n+s}(x) \right]^2.
\]

**Proof:** (Sometimes, instead of \( T_n(x) \), we shall write plainly \( T_n \).)

By setting \( x = \cos \theta \) (and so \( T_n = \cos n \theta \)), one has

\[
T_n T_{n+r+s} = \cos n \theta \cos(n+r+s) \theta = \frac{1}{2} \left[ \cos(2n+r+s) \theta + \cos(r+s) \theta \right]
\]

and

\[
T_{n+r} T_{n+s} = \cos(n+r) \theta \cos(n+s) \theta = \frac{1}{2} \left[ \cos(2n+r+s) \theta + \cos(r-s) \theta \right]
\]
and, consequently,

\[ T_n T_{n+r+s} - T_{n+r} T_{n+s} = \frac{1}{2} \left[ \cos(r + s) \theta - \cos(r - s) \theta \right] . \]

Hence,

\[ T_n T_{n+r+s} + \frac{1}{2} (T_{r-s} - T_{r+s}) = T_{n+r} T_{n+s}, \]

which proves (2.1).

One has clearly

\[ \frac{1}{4} (T_{r-s} - T_{r+s})^2 = T_n^2 T_{n+s}^2 + T_n^2 T_{n+r+s}^2 - 2 T_n T_{n+r} T_{n+s} T_{n+r+s} \]

and so

\[ 4 T_n T_{n+r} T_{n+s} T_{n+r+s} + \frac{1}{4} (T_{r-s} - T_{r+s})^2 = (T_n T_{n+r+s} + T_{n+r} T_{n+s})^2, \]

which proves (2.2).

Now, we are going to state the following

**Theorem 1.** If \((T_n)_{n \geq 0}\) is the sequence of Chebyshev polynomials of the first kind, then the product of any two distinct elements of the set

\[ \{T_n, T_{n+2r}, T_{n+4r}, 4T_{n+r} T_{n+2r} T_{n+3r}\}, \quad n, r \in \mathbb{N}, \]

increased by \(\left[\frac{1}{2} (T_h - T_k)^4\right]\), where \(T_h\) and \(T_k\), with \(k > h \geq 0\), are suitable terms of the sequence \((T_n)_{n \geq 0}\), and \(t\) is 1 or 2, according to the number of factors \(T\), in that product, is 2 or 4, is a perfect square.

**Proof:** Indeed, if one sets \(s = r\), in (2.1), one obtains

\[ T_n T_{n+2r} + \frac{1}{2} (T_0 - T_{2r}) = T_{n+r}^2. \]  

If \(r\) is replaced by \(2r\), in (2.3), one gets

\[ T_n T_{n+4r} + \frac{1}{2} (T_0 - T_{4r}) = T_{n+2r}^2. \]  

By replacing, in (2.3) \(n\) by \(n + 2r\), one obtains

\[ T_{n+2r} T_{n+4r} + \frac{1}{2} (T_0 - T_{2r}) = T_{n+3r}^2. \]
If one puts $s = 2r$, in (2.2), one gets

\[(2.6)\quad 4T_n T_{n+r} T_{n+2r} T_{n+3r} + \left[\frac{1}{2}(T_r - T_{3r})\right]^2 = (T_n T_{n+3r} + T_{n+r} T_{n+2r})^2.
\]

Now, by changing $n$ into $n + r$, in (2.6), it comes

\[(2.7)\quad 4T_{n+r} T_{n+2r} T_{n+3r} T_{n+4r} + \left[\frac{1}{2}(T_r - T_{3r})\right]^2 = (T_{n+r} T_{n+4r} + T_{n+2r} T_{n+3r})^2.
\]

If one replaces $n$ by $n + r$, in (2.2), and, furthermore, one puts $s = r$, one obtains

\[(2.8)\quad 4T_{n+r} T_{n+2r}^2 T_{n+3r} + \left[\frac{1}{2}(T_r - T_{2r})\right]^2 = (T_{n+r} T_{n+3r} + T_{n+2r}^2)^2,
\]

which completes the proof of the theorem above.

### 3. Applications to the Fibonacci numbers

There is a connection between, on the one hand, the sequence of Fibonacci numbers, $(F_n)_{n \geq 0}$, with

\[(3.1)\quad F_n = \frac{1}{\sqrt{5}} \left[ (\frac{1 + \sqrt{5}}{2})^n - (\frac{1 - \sqrt{5}}{2})^n \right],
\]

and, on the other hand, the sequences $(U_n)_{n \geq 0}$ and $(T_n)_{n \geq 0}$.

Indeed, from (1.5), it results

\[(3.2)\quad U_n \left(\frac{i}{2}\right) = \frac{1}{i\sqrt{5}} \left[ (\frac{i}{2} + \frac{i}{2} \sqrt{5})^{n+1} - (\frac{i}{2} - \frac{i}{2} \sqrt{5})^{n+1} \right] = i^n F_{n+1}.
\]

Now, from (1.8) and (3.2), one finds

\[T_n \left(\frac{i}{2}\right) = U_n \left(\frac{i}{2}\right) - \frac{i}{2} U_{n-1} \left(\frac{i}{2}\right) = \frac{i^n}{2} (2F_{n+1} - F_{n})
\]

and, since $F_{n+1} = F_n + F_{n-1}$, one has

\[(3.3)\quad T_n \left(\frac{i}{2}\right) = \frac{i^n}{2} (F_n + 2F_{n-1}).
\]

Thus, from (3.3) and (3.3), it follows

\[
\frac{i^n}{2} (F_n + 2F_{n-1}) \cdot \frac{i^{n+2r}}{2} (F_{n+2r} + 2F_{n+2r-1}) + \frac{1}{2} \left[ 1 - \frac{i^{2r}}{2} (F_{2r} + 2F_{2r-1}) \right] = \\
= \left[ \frac{i^{n+r}}{2} (F_{n+r} + 2F_{n+r-1}) \right]^2,
\]
that is to say,

\[ (-1)^{n+r} \left( \frac{1}{2} F_n F_{n+2r} + \frac{1}{2} F_n F_{n+2r-1} + \frac{1}{2} F_{n-1} F_{n+2r} + F_{n-1} F_{n+2r-1} \right) - \frac{1}{2} \left[ (-1)^r \left( \frac{1}{2} F_{2r} + F_{2r-1} \right) - 1 \right] = (-1)^{n+r} \left( \frac{1}{2} F_{n+1} + F_{n+r-1} \right)^2, \]

and hence,

\[ F_n F_{n+2r} + 2F_{n+1} F_{n+2r-1} + 2F_{n-1} F_{n+2r+1} + 2(-1)^{n+r} - (-1)^n \left( F_{2r} + 2F_{2r-1} \right) = F_{n+r}^2 + 4F_{n+r-1} F_{n+r+1}, \]

and, analogously from the relations (2.4)–(2.8) and (3.3) other equalities can be obtained.

Other more interesting results can be obtained by making use of another connection between \( T_n \) and \( F_n \). In fact, from (1.4), it results

\[ T_n \left( \frac{i}{2} \right) = i^n \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n + \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] \]

\[ = i^n \left\{ \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{2n} - \left( \frac{1 - \sqrt{5}}{2} \right)^{2n} \right] / \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] \right\}, \]

and hence, for \( n > 0 \),

\[ T_n \left( \frac{i}{2} \right) = \frac{i^n}{2} \cdot \frac{F_{2n}}{F_n}. \]

Thus, from (2.3) and (3.5), one obtains

\[ \frac{i^{2n+2r}}{4} \cdot \frac{F_{2n}}{F_n} \cdot \frac{F_{2n+4r}}{F_{n+2r}} + \frac{1}{2} \left( 1 - \frac{i^{2r} F_{4r}}{2 F_{2r}} \right) = \frac{i^{2n+2r}}{4} \left( \frac{F_{2n+2r}}{F_{n+r}} \right)^2, \]

whence,

\[ \frac{F_{2n}}{F_n} \cdot \frac{F_{2n+4r}}{F_{n+2r}} + (-1)^n \left[ 2(-1)^r - \frac{F_{4r}}{2 F_{2r}} \right] = \left( \frac{F_{2n+2r}}{F_{n+r}} \right)^2. \]

Analogously, from (2.4) and (3.5), it follows that

\[ \frac{i^{2n+4r}}{4} \cdot \frac{F_{2n}}{F_n} \cdot \frac{F_{2n+8r}}{F_{n+4r}} + \frac{1}{4} \left( 2 - i^{4r} \frac{F_{8r}}{F_{4r}} \right) = \left( \frac{i^{n+r}}{2} \cdot \frac{F_{2n+4r}}{F_{n+2r}} \right)^2, \]

and, consequently, one has

\[ \frac{F_{2n}}{F_n} \cdot \frac{F_{2n+8r}}{F_{n+4r}} + (-1)^n \left[ 2 - \frac{F_{8r}}{F_{4r}} \right] = \left( \frac{F_{2n+4r}}{F_{n+2r}} \right)^2. \]
By using (2.5) and (3.5), one obtains

\[(3.8) \quad \frac{F_{2n+4r}}{F_{n+2r}} \cdot \frac{F_{2n+8r}}{F_{n+4r}} + (-1)^n \left( 2(-1)^r - \frac{F_{4r}}{F_{2r}} \right) = \left( \frac{F_{2n+6r}}{F_{n+3r}} \right)^2. \]

from (2.6) and (3.5), it results

\[(3.9) \quad 4 \cdot \frac{F_{2n}}{F_n} \cdot \frac{F_{2n+2r}}{F_{n+r}} \cdot \frac{F_{2n+4r}}{F_{n+2r}} \cdot \frac{F_{2n+6r}}{F_{n+3r}} + \left( \frac{F_{2r}}{F_r} - (-1)^r \frac{F_{6r}}{F_{3r}} \right)^2 = \left( \frac{F_{2n}}{F_n} \cdot \frac{F_{2n+6r}}{F_{n+3r}} + \frac{F_{2n+2r}}{F_{n+r}} \cdot \frac{F_{2n+4r}}{F_{n+2r}} \right)^2. \]

from (2.7) and (3.5), one obtains

\[(3.10) \quad 4 \cdot \frac{F_{2n+2r}}{F_{n+r}} \cdot \frac{F_{2n+4r}}{F_{n+2r}} \cdot \frac{F_{2n+6r}}{F_{n+3r}} \cdot \frac{F_{2n+8r}}{F_{n+4r}} + \left( \frac{F_{2r}}{F_r} - (-1)^r \frac{F_{6r}}{F_{3r}} \right)^2 = \left( \frac{F_{2n+2r}}{F_{n+r}} \cdot \frac{F_{2n+8r}}{F_{n+4r}} + \frac{F_{2n+4r}}{F_{n+2r}} \cdot \frac{F_{2n+6r}}{F_{n+3r}} \right)^2. \]

Finally, from (2.8) and (3.5), it results

\[(3.11) \quad 4 \cdot \frac{F_{2n+2r}}{F_{n+r}} \left( \frac{F_{2n+4r}}{F_{n+2r}} \right)^2 \cdot \frac{F_{2n+6r}}{F_{n+3r}} + \left( 2 - (-1)^r \frac{F_{4r}}{F_{2r}} \right)^2 = \left[ \frac{F_{2n+2r}}{F_{n+r}} \cdot \frac{F_{2n+6r}}{F_{n+3r}} + \left( \frac{F_{2n+4r}}{F_{n+2r}} \right)^2 \right]^2. \]

This means that the following holds:

**Theorem 2.** If \((F_n)_{n \geq 0}\) is the sequence of Fibonacci numbers, then the product of any two distinct elements of the set

\[(3.12) \quad \left\{ \frac{F_{2n}}{F_n}, \frac{F_{2n+4r}}{F_{n+2r}}, \frac{F_{2n+8r}}{F_{n+4r}} \cdot 4 \cdot \frac{F_{2n+2r}}{F_{n+r}} \cdot \frac{F_{2n+6r}}{F_{n+3r}} \right\}, \quad \text{with } n > 0,
\]

increased by \(\pm (\pm 2 - \frac{F_{3h}}{F_{2h}})\), if only 2 factors occur in that product; increased by \((\frac{F_{3h}}{F_{2h}} - \frac{F_{5h}}{F_{4h}})^2\), if 4 different factors occur in that product, and increased by \((2 \pm \frac{F_{3h}}{F_{2h}})^2\), if 4 factors occur in that product, but only three are different; \(h\) is the difference between the greatest and the least subscripts of \(F\) in the denominators of the factors and \(l\) is the difference between the subscripts of \(F\) in the denominators of the intermediate factors.

It is clear that the four integers belonging to the set (3.12) are not necessarily Fibonacci numbers and so the set (3.12) is a Diophantine quadruple, but, in general, it is not a Fibonacci quadruple.
In pursuance of a suggestion of the referee, we are going to present the results contained in Theorem 2, under another form, through the introduction of the Lucas numbers.

In [2], p. 395, L.E. Dickson says that E. Lucas

“employed the roots \( a, b \) of \( x^2 = x + 1 \) and set

\[
\begin{align*}
  u_n &= \frac{a^n - b^n}{a - b}, \\
  v_n &= a^n + b^n = \frac{u_{2n}}{u_n} = u_{n-1} + u_{n+1}.
\end{align*}
\]

The \( u \)'s form the series of Pisano [Fibonacci] with terms 0, 1 prefixed, so that

\[
u_0 = 0, \quad u_1 = u_2 = 1, \quad u_3 = 2.\]

The \( v \)'s are the Lucas numbers.

One has \( v_n = u_{n-1} + u_{n+1} \). In fact, the equality

\[
a^n + b^n = \frac{a^{n-1} - b^{n-1}}{a - b} + \frac{a^{n+1} - b^{n+1}}{a - b}
\]

is equivalent to

\[
a^{n+1} - b^{n+1} + a b^n - a^n b = a^{n-1} - b^{n-1} + a^{n+1} - b^{n+1}
\]

and this is equivalent to

\[
a b(b^{n-1} - a^{n-1}) = a^{n-1} - b^{n-1},
\]

which is true, since \( ab = -1 \).

One has also

\[
v_n = a^n + b^n = \frac{(a^{2n} - b^{2n})/(a - b)}{(a^n - b^n)/(a - b)} = \frac{u_{2n}}{u_n} = \frac{F_{2n}}{F_n} = L_n,
\]

with \( n > 0 \).

Thus, by taking into account Theorem 2, one concludes that the following holds:

**Theorem 2'.** If \( (L_n)_{n>0} \) is the sequence of the Lucas numbers, then the product of any two distinct elements of the set

\[
\{L_n, L_{n+2r}, L_{n+4r}, 4L_{n+r}L_{n+2r}L_{n+3r}\}, \quad \text{with} \quad n > 0,
\]

increased by \( \pm(\pm 2 - L_h) \), if only 2 factors \( L \) occur in that product; increased by \( (L_k - L_h)^2 \), if 4 different factors \( L \) occur in that product, and increased by \( (2 \pm L_h)^2 \), if 4 factors \( L \) occur in that products, but only three are different; \( h \) is the difference between the greatest and the least subscripts of \( L \), and \( k \) is the difference between the subscripts of \( L \) in the intermediate factors.
4 – A generalization of the Chebyshev polynomials of the first and the second kind

Let us consider the sequence of polynomials \((S_n(x))_{n \geq 0}\) defined by the recurrence relation

\[
S_{n+2}(x) = 2x S_{n+1}(x) - S_n(x), \quad n \geq 0,
\]

with \(S_0(x) = a\) and \(S_1(x) = b\), being \(a, b \in \mathbb{Z}[x]\).

Let \(g(y) = S_0(x) + S_1(x) y + ... + S_n(x) y^n + ...\), be the generating function of the sequence \((S_n(x))_{n \geq 0}\). By making use of the reducing polynomial, \(k(y) = 1 - 2xy + y^2\), one obtains the following finite form for \(g(y)\):

\[
g(y) = \frac{a + (b - 2ax) y}{1 - 2xy + y^2},
\]

which can be written as

\[
g(y) = \frac{A}{y - (x + \sqrt{x^2 - 1})} + \frac{B}{y - (x - \sqrt{x^2 - 1})},
\]

with

\[
A = \frac{(b - 2ax) \sqrt{x^2 - 1} + a + (b - 2ax) x}{2\sqrt{x^2 - 1}},
\]

\[
B = \frac{(b - 2ax) \sqrt{x^2 - 1} - [a + (b - 2ax) x]}{2\sqrt{x^2 - 1}},
\]

where \(x \neq \pm 1\).

By operating as in §1 in order to get the formula (1.4), one obtains

\[
S_n(x) = \left(\frac{a}{2} + \frac{ax - b}{2\sqrt{x^2 - 1}}\right) \left(x - \sqrt{x^2 - 1}\right)^n + \left(\frac{a}{2} - \frac{ax - b}{2\sqrt{x^2 - 1}}\right) \left(x + \sqrt{x^2 - 1}\right)^n.
\]

One sees that, for \(a = 1\) and \(b = x\), one has \(S_n(x) = T_n(x)\) and, for \(a = 1\) and \(b = 2x\), one has \(S_n(x) = U_n(x)\).

It follows also immediately that, if one sets \(x = \cos \theta\) in (4.2), then

\[
S_n(\cos \theta) = a \cos n \theta - \frac{(a \cos \theta - b) \sin n \theta}{\sin \theta}.
\]

If one puts \(a = 1\) and \(b = x = \cos \theta\), one obtains

\[
S_n(\cos \theta) = \cos n \theta = T_n(\cos \theta),
\]
and, for \( a = 1 \) and \( b = 2x = 2\cos \theta \), one obtains
\[
S_n(\cos \theta) = \cos n \theta - \frac{-\cos \theta \sin n \theta}{\sin \theta} = \frac{\sin(n + 1) \theta}{\sin \theta} = U_n(\cos \theta),
\]
as was to be expected.

If \( a = T_j(x) \) and \( b = T_{j+1}(x) \), then one has:
\[
S_n(x) = S_n(\cos \theta) = \cos j \theta \cos n \theta - \left[ \cos j \theta \cos \theta - \cos(j + 1) \theta \right] \sin n \theta \sin \theta
\]
\[
= \cos(j + n) \theta + \sin j \theta \sin n \theta - \frac{\sin j \theta \sin \theta \sin n \theta}{\sin \theta}
\]
\[
= \cos(j + n) \theta = T_{j+n}(x).
\]

If \( a = U_j(x) \) and \( b = U_{j+1}(x) \), then one has, by (4.3),
\[
S_n(x) = S_n(\cos \theta) = \frac{\sin(j + 1) \theta \cos n \theta}{\sin \theta} - \left[ \sin(j + 1) \theta \cos \theta - \sin(j + 2) \theta \right] \sin n \theta \sin \theta
\]
\[
= \frac{\sin(j + 1) \theta \cos n \theta}{\sin \theta} + \frac{\cos(j + 1) \theta \sin n \theta}{\sin \theta} = \frac{\sin(j + n + 1) \theta}{\sin \theta}
\]
\[
= U_{j+n}(\cos \theta) = U_{j+n}(x).
\]

Now, we are going to prove, for \( S_n (= S_n(x) = S_n(\cos \theta)) \), a result analogous to Lemma 1.

**Lemma 2.** If \((S_n)_{n \geq 0}\) is the sequence of polynomials defined by (4.1), then one has:

\[
(4.4) \quad S_n S_{n+r+s} + \frac{1}{2} \cdot \frac{a^2 + b^2 - 2abx}{1-x^2} (T_r - s) = S_{n+r} S_{n+s}
\]
and

\[
(4.5) \quad 4S_n S_{n+r} S_{n+s} S_{n+r+s} + \left[ \frac{1}{2} \cdot \frac{a^2 + b^2 - 2abx}{1-x^2} (T_r - s) \right]^2
\]
\[
= (S_n S_{n+r+s} + S_{n+r} S_{n+s})^2.
\]
Proof: Indeed, by taking into account the relation (4.3), one has

\[ S_n(\cos \theta) S_{n+r+s}(\cos \theta) - S_{n+r}(\cos \theta) S_{n+s}(\cos \theta) = \]

\[ = \left( a \cos n \theta - \frac{a \cos \theta - b}{\sin \theta} \sin n \theta \right) \left[ a \cos (n+r+s) \theta - \frac{a \cos \theta - b}{\sin \theta} \sin (n+r+s) \theta \right] - \]

\[ - \left[ a \cos (n+r) \theta - \frac{a \cos \theta - b}{\sin \theta} \sin (n+r) \theta \right] \left[ a \cos (n+s) \theta - \frac{a \cos \theta - b}{\sin \theta} \sin (n+s) \theta \right] = \]

\[ = a^2 \left[ \cos n \theta \cos (n + r + s) \theta - \cos (n + r) \theta \cos (n + s) \theta \right] + \]

\[ + \left( \frac{a \cos \theta - b}{\sin \theta} \right)^2 \left[ \sin n \theta \sin (n + r + s) \theta - \sin (n + r) \theta \sin (n + s) \theta \right] + \]

\[ + a \cdot \frac{a \cos \theta - b}{\sin \theta} \left\{ \left[ \sin (n + r) \theta \cos (n + s) \theta + \cos (n + r) \theta \sin (n + s) \theta \right] \right\} = \]

\[ = a^2 \left[ \cos (2n + r + s) \theta + \cos (r + s) \theta \right] - \frac{\cos (2n + r + s) \theta + \cos (r - s) \theta}{2} - \]

\[ + \left( \frac{a \cos \theta - b}{\sin \theta} \right)^2 \left[ \frac{\cos (r + s) \theta - \cos (r - s) \theta}{2} - \frac{\cos (r + s) \theta - \cos (r - s) \theta}{2} \right] = \]

\[ = \frac{1}{2} \left[ a^2 + \left( \frac{a \cos \theta - b}{\sin \theta} \right)^2 \right] \left[ \cos (r + s) \theta - \cos (r - s) \theta \right] = \]

\[ \frac{1}{2} \cdot \frac{a^2 + b^2 - 2 a b \cos \theta}{\sin^2 \theta} \cdot \left[ \cos (r + s) \theta - \cos (r - s) \theta \right], \]

and, consequently, since \( T_j(x) = \cos jx \), one has (4.4).

Now, from (4.4), one obtains

\[ 4 S_n S_{n+r} S_{n+s} S_{n+r+s} + \left\{ \frac{1}{2} : \frac{a^2 + b^2 - 2 a b \cos \theta}{\sin^2 \theta} \left[ \cos (r - s) \theta - \cos (r + s) \theta \right] \right\}^2 = \]

\[ = 4 S_n S_{n+r} S_{n+s} S_{n+r+s} + (S_n S_{n+r+s} - S_{n+r} S_{n+s})^2 = \]

\[ = (S_n S_{n+r+s} + S_{n+r} S_{n+s})^2, \]

as desired. ■

It is clear that (4.4) generalizes (2.1); in fact, for \( a = 1 \) and \( b = x \), one obtains \( S_0(x) = T_0(x) = 1 \) and \( S_1(x) = x = \cos \theta = T_1(x) \), and (4.4) becomes (2.1). One sees also that (4.4) generalizes the identity

\[ U_n(x) U_{n+r+s}(x) + U_{r-1}(x) U_{s-1}(x) = U_{n+r}(x) U_{n+s}(x), \]
obtained by Gheorghe Udrea for the Chebyshev polynomials of the second kind, in [12]; in fact, for $a = 1$ and $b = 2x = 2\cos \theta$, the identity (4.4) becomes the identity above, obtained in [12].

Now, one can state the following

**Theorem 3.** Let $(S_n(x))_{n \geq 0}$ be the sequence of polynomials defined by the recurrence relation

$$S_{n+2}(x) = 2x S_{n+1}(x) - S_n(x), \quad n \geq 0,$$

with $S_0(x) = a$, $S_1(x) = b$, $x \in \mathbb{C}$ and $x \neq \pm 1$, where $a, b \in \mathbb{Z}[x]$. Then, the product of any two distinct elements of the set

$$(4.6) \quad \left\{ S_n(x), S_{n+2r}(x), S_{n+4r}(x), 4S_{n+r}(x)S_{n+2r}(x)S_{n+3r}(x) \right\},$$

increased by

$$\left[ \frac{1}{2} \cdot \frac{a^2 + b^2 - 2abx}{1 - x^2} \right]^t (T_h(x) - T_k(x)),
$$

where $T_h$ and $T_k$ are suitable terms of the sequence $(T_n)_{n \geq 0}$, independent of $n$, with $h < k$, and $t = 1$ or $t = 2$, according to the number of factors $S$, in that product, is 2 or 4, is a perfect square.

**Proof:** One proceeds as in the proof of Theorem 1.

Thus, by setting $s = r$, in (4.4), one obtains

$$S_n \cos(\theta) S_{n+2r}(\cos \theta) + \frac{1}{2} \cdot \frac{a^2 + b^2 - 2abx}{1 - x^2} \left( 1 - \cos 2r\theta \right) = \left( S_{n+r}(\cos \theta) \right)^2,$$

that is to say,

$$(4.7) \quad S_n(x) S_{n+2r}(x) + \frac{1}{2} \cdot \frac{a^2 + b^2 - 2abx}{1 - x^2} (T_0(x) - T_{2r}(x)) = \left( S_{n+r}(x) \right)^2.
$$

By replacing $r$ by $2r$, in (4.7), one gets

$$(4.8) \quad S_n(x) S_{n+4r}(x) + \frac{1}{2} \cdot \frac{a^2 + b^2 - 2abx}{1 - x^2} (T_0(x) - T_{4r}(x)) = \left( S_{n+r}(x) \right)^2.
$$

By replacing $n$ by $n + 2r$, in (4.7), it results in

$$(4.9) \quad S_{n+2r}(x) S_{n+4r}(x) + \frac{1}{2} \cdot \frac{a^2 + b^2 - 2abx}{1 - x^2} (T_0(x) - T_{2r}(x)) = \left( S_{n+3r}(x) \right)^2.$$
By setting $s = 2r$, in (4.5), one gets

$$4S_n(x)S_{n+r}(x)S_{n+2r}(x)S_{n+3r}(x) + \left\{ \frac{1}{2} \left[ \frac{a^2 + b^2 - 2abx}{1-x^2} \right] \right\}^2 = [S_n(x)S_{n+3r}(x) + S_{n+r}(x)S_{n+2r}(x)]^2.$$  

(4.10)

If one replaces $n$ by $n + r$, in (4.10) one obtains

$$4S_{n+r}(x)S_{n+2r}(x)S_{n+3r}(x)S_{n+4r}(x) + \left\{ \frac{1}{2} \left[ \frac{a^2 + b^2 - 2abx}{1-x^2} \right] \right\}^2 = [S_{n+r}(x)S_{n+4r}(x) + S_{n+2r}(x)S_{n+3r}(x)]^2.$$  

(4.11)

Finally, by setting $s = r$, in (4.5), it results

$$4S_n(x) (S_{n+r}(x))^2S_{n+2r}(x) + \left\{ \frac{1}{2} \left[ \frac{a^2 + b^2 - 2abx}{1-x^2} \right] \right\}^2 = [S_n(x)S_{n+2r}(x) + (S_{n+r}(x))^2]^2,$$

(4.12)

thus completing the proof. □

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José Morgado,
Centro de Matemática da Faculdade de Ciências da Universidade do Porto,
Porto – PORTUGAL