CENTRAL MORPHISMS AND ENVELOPES OF HOLOMORPHY

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Abstract: In this paper we study a particular class of continuous algebra morphisms, the so-called $C$-central $A$-morphisms; i.e. continuous $A$-morphisms between topological $A$-algebras (viz. we take coefficients from a topological algebra $A$) such that their images have $C$-trivial center. In particular, we examine such morphisms for algebra-valued holomorphic functions on a complex manifold $X$, giving conditions that the set of the previous morphisms be the classical envelope of holomorphy of $X$.

1 – Introduction

The envelope of holomorphy of a domain $X \subseteq \mathbb{C}^n$, $\text{Env}_{O(X)}(X)$, is classically defined as the maximal Riemann domain to which every holomorphic $\mathbb{C}$-valued function on $X$ can be extended [8]. Moreover, $\text{Env}_{O(X)}(X)$ is equal to the spectrum (Gel’fand space) of $O(X)$, denoted by $\text{M}(O(X))$ [6], i.e. the set of continuous $\mathbb{C}$-characters of the algebra $O(X)$ of $\mathbb{C}$-valued holomorphic functions on $X$, the latter algebra being endowed with the compact open topology. In this context, one is interested in the following: Given a complex (analytic) manifold $X$ and a locally $m$-convex algebra $A$ (not necessarily commutative), which is the maximal complex manifold $Y$ such that every $A$-valued holomorphic function on $X$ can be extended to $Y$? An answer was given by J.G. Craw [1] in the case of a Riemann domain $X$ and a Banach algebra $A$, considering as $Y$ the set of $A$-characters of $O(X, A)$ (viz. $A$-morphisms of $O(X, A)$ into $A$) which restrict to a character of $O(X)$. In [4] (see also [5]) we gave another interpretation of the previous result by realizing it as the set of $C$-central $A$-morphisms of $O(X, A)$ into a locally

Received: January 28, 1994.
1980 Mathematics Subject Classification: Primary 46M05, 32E15, 32E25; Secondary 46M40, 32E10.

Keywords and Phrases: Central morphisms, envelopes of holomorphy, Runge pairs, inductive limits, tensor products, spectra.
m-convex algebra (cf. §3 for definitions). This set is the \( \text{Env}_{O(X,A)}(X) \) (envelope of \( X \) with respect to \( O(X,A) \)), in analogy with the complex case; indeed, one can prove that it is a Riemann domain and \( O(X,A) \)-convex, when \( X \) is a Riemann domain and \( A \) a Fréchet locally m-convex algebra (ibid.).

We consider below, in an entirely general context, continuous \( A \)-morphisms (§2) and \( C \)-central \( A \)-morphisms (§3) with respect to topological tensor product algebras. Precisely, using the above technique, we study these sets of morphisms defined on algebra-valued function algebras, since these algebras can be expressed as tensor product algebras under suitable conditions (cf. (2.5), (2.6)). Thus, we take analogous results to the classical case of the \( C \)-valued algebras through the corresponding spectra (cf. Corollaries 3.2, 3.3). In particular, we examine in §4 the set of \( C \)-central \( A \)-morphisms on \( O(X,A) \), and give conditions under which one identifies the set \( \text{Env}_{O(X,A)}(X) \) with the \( \text{Env}_{O(X)}(X) \) (cf. Corollary 4.1). To this end we use the notion of a Runge pair with respect to the sheaf \( O \otimes A \) (cf. also [7]).

2 – The generalized \( A \)-spectrum of \( E \otimes F \)

Given a topological algebra \( A \), an \( A \)-algebra \( E \) is called a topological \( A \)-algebra if \( E \) is a topological algebra and the “action” of \( A \) on \( E \) is a (jointly) continuous map.

If \( E, F \) are topological \( A \)-algebras, the generalized \( A \)-spectrum of \( E \) with respect to (w.r.t.) \( F \) is the set \( M_A(E,F) \) of non-zero continuous \( A \)-morphisms of \( E \) into \( F \), equipped with the topology induced on it by \( \mathcal{L}_A(E,F)_s \) (the space of continuous \( A \)-linear maps between the corresponding modules with the simple convergence topology on \( E \); cf. [2: §3]). If the algebras involved have identities, the respective morphisms are assumed to be “identity preserving”.

Now, given a (\( C \)-) algebra \( E \) and an \( A \)-algebra \( F \), we consider the corresponding (algebraic) tensor product algebra \( E \otimes F \), which is an \( A \)-algebra such that

\[
(2.1) \quad a \cdot (x \otimes y) := x \otimes a \cdot y,
\]

for any \( a \in A \) and \( x \otimes y \in E \otimes F \). Analogously, if \( E \) is an \( A \)-algebra and \( F \) a (\( C \)-) algebra.

**Definition 2.1.** Let \( E, F \) be topological (\( C \)-) algebras with \( F \) being a topological \( A \)-algebra. By a compatible topology on the corresponding tensor
product \( A \)-algebra \( E \otimes F \) we mean a (Hausdorff) topology \( \tau \) such that the pair \( (E \otimes F, \tau) \equiv E \otimes F \) is a topological \( A \)-algebra of the same type as \( E, F \).

This type of compatibility of a tensorial topology is analogous to that of [6: Chapter X, Definition 3.1] and [2: Definition 1.1] suitably modified in our case. In the sequel, we are interested in compatible topologies \( \tau \) satisfying the following conditions

(2.2) The canonical map of \( E \times F \) into \( E \otimes F \) is separately continuous.

(2.3) For every topological \( A \)-algebra \( G \), and for any pair \((f, g) \in M_{C}(E, G) \times M_{A}(F, G)\) one has \( f \otimes g \in L_{A}(E \otimes F, G) \).

Here by \( M_{C}(E, G) \) we consider the generalized \( C \)-spectrum of \( E \) (w.r.t.) \( G \) and moreover \( f \otimes g \) is defined by \((f \otimes g)(x \otimes y) := f(x) \cdot g(y), x \otimes y \in E \otimes F \).

A stronger version of (2.3) is applied when one has to consider completed tensor product \( A \)-algebras. That is, we assume that

(2.4) For any equicontinuous subsets \( M \subseteq M_{C}(E, G), N \subseteq M_{A}(F, G) \), the set \( M \otimes N \equiv \{ f \otimes g : f \in M, g \in N \} \) is an equicontinuous subset of \( L_{A}(E \otimes F, G) \).

Examples. In case of locally convex algebras the projective (\( C \))-tensorial topology \( \pi \) is a compatible topology on \( E \otimes F \) (Definition 2.1) satisfying (2.2) and (2.4) (and hence (2.3), cf. [6: Chapter X, Lemma 3.1]). The remark is still in force concerning locally convex \( A \)-algebras with continuous multiplication, or yet in case of locally \( m \)-convex ones. On the other hand, the preceding is still possible within the context of not necessarily locally convex \( A \)-algebras. Thus, if \( A \) is a locally bounded algebra with continuous multiplication [6], then every locally bounded \( A \)-algebra with continuous multiplication is a topological \( A \)-algebra (not necessarily locally convex). So for a pair \((E, F)\) of locally bounded algebras with \( F \) being a locally bounded \( A \)-algebra, the corresponding compatible topology on \( E \otimes F \) (cf. [6: Chapter VI, Theorem 3.1]) is, in fact, a (not necessarily locally convex) topology on \( E \otimes F \), as above.

One has the above situation taking algebra-valued function algebras which may be considered as topological tensor product \( A \)-algebras in the sense of the previous remarks.

Thus, if \( X \) is a completely regular \( k \)-space and \( E \) a complete locally convex
\textbf{A}-algebra, then one has

\begin{equation}
C_c(X, E) = C_c(X) \hat{\otimes} E
\end{equation}

within an isomorphism of locally convex \textbf{A}-algebras (cf. [6: Chapter XI, Theorem 1.1]). Here \(C_c(X, E)\) (resp. \(C_c(X)\)) is the algebra of \(E\)- (resp. \(C\)-) valued continuous functions on \(X\), equipped with the compact-open topology, which is also a locally convex \textbf{A}-algebra defining \((af)(x) := a \cdot f(x)\), for every \(a \in \textbf{A}\), \(f \in C_c(X, E)\), \(x \in E\). In the second member of (2.5) \(\varepsilon\) denotes the biprojective tensorial topology (ibid.), which is also a compatible topology as above through the isomorphism (2.5). One has analogous results by considering \(E\)-valued holomorphic (resp. \(C^\infty\)-) functions. Precisely, if \(K\) is a compact subset of a second countable complex manifold \(X\) (resp. a paracompact \(C^\infty\)-manifold \(X\)) and \(E\) a locally convex \textbf{A}-algebra, then one gets the following isomorphisms of locally convex \textbf{A}-algebras

\begin{equation}
O(K, E) = O(K) \hat{\otimes} E \quad (\text{resp. } C^\infty(X, E)) = C^\infty(X) \hat{\otimes} E
\end{equation}

(cf. §4 and also [6: Chapter XI, Lemma 4.1, Theorem 2.1, (2.8)]). For the relevant definitions cf. [6].

\textbf{Proposition 2.1.} Let \(E, F\) be unital topological algebras with \(F\) being also a topological \textbf{A}-algebra and \(G\) a unital topological \textbf{A}-algebra with continuous multiplication. Moreover, let \(\tau\) be a compatible topology on \(E \otimes F\) satisfying (2.2), (2.3) and the closed set

\begin{equation}
Q = \{(f, g) : f(x)g(y) = g(y)f(x); \ x \in E, \ y \in F\} \subseteq M_C(E, G) \times M_A(F, G).
\end{equation}

Then one gets the next homeomorphism

\begin{equation}
M_A(E \hat{\otimes}_\tau F, G) = Q
\end{equation}

and for \(G\) commutative

\begin{equation}
M_A(E \hat{\otimes}_\tau F, G) = M_C(E, G) \times M_A(F, G).
\end{equation}

\textbf{Proof:} For any \(h \in M_A(E \hat{\otimes}_\tau F, G)\) we define

\begin{equation}
f(x) := h(x \otimes 1_F), \ x \in E, \quad \text{and} \quad g(y) := h(1_E \otimes y), \ y \in F,
\end{equation}

\(\square\)
with $1_E$, $1_F$ the identities of $E$, $F$ respectively, such that one has

$$(2.11) \quad h = f \otimes g$$

and moreover, $(f, g) \in M_C(E, G) \times M_A(F, G)$; hence one gets the map

$$(2.12) \quad M_A(E \otimes \tau F, G) \rightarrow M_C(E, G) \times M_A(F, G),$$

which is an injection (cf. (2.11)). Furthermore for a given $(f, g) \in Q$ one has $h := f \otimes g \in M_A(E \otimes \tau F, G)$ and every element thus defined yields according to (2.11) the initial pair $(f, g)$, that is (2.12) is a bijection onto $Q$. The bicontinuity of (2.12) can be proved analogously to [6: Chapter XII, Lemma 3.1]. Concerning (2.9), this is immediate from (2.7), (2.8). 

For the study of the generalized spectrum of the (complete) topological $A$-algebra $E \otimes \tau F$ we need some more terminology.

Thus, let $E$, $G$ be topological $A$-algebras where $E$ has continuous multiplication and $G$ is complete. The continuous bijection

$$(2.13) \quad M_A(E, G) \rightarrow M_A(\hat{E}, G): f \mapsto \overline{f},$$

where $\overline{f}$ is the (continuous) extension of $f$ to the completion $\hat{E}$ of $E$, is a homeomorphism iff either one of the sets $M_A(E, G)$, $M_A(\hat{E}, G)$ is locally equicontinuous (cf. [2: (3.10), (3.11)]).

Proposition 2.1 and relation (2.13) yield the following lemma, whose proof is analogous to [6: Chapter VI, Lemma 6.2].

**Lemma 2.1.** Let $E$, $F$, $G$ be topological algebras as in Proposition 2.1 and $\tau$ a compatible topology satisfying (2.2), (2.3). Moreover, consider the following assertions

i) $M_C(E, G)$, $M_A(F, G)$ are both locally equicontinuous;

ii) $M_A(E \otimes \tau F, G)$ is locally equicontinuous.

Then i)$\Rightarrow$ii). Besides, or every $(f, g) \in Q$, there exist an equicontinuous neighbourhood $U$ of $f$ in $M_C(E, G)$ and $V$ of $g$ in $M_A(F, G)$ such that $U \otimes V$ is an equicontinuous neighbourhood of $f \otimes g$ in $M_A(E \otimes \tau F, G)$. In particular ii)$\Rightarrow$i) as well, whenever $G$ is commutative. 

Now, we have the main result of this section as follows.
Theorem 2.1. Let \( E, F \) be unital topological algebras with continuous multiplication where \( F \) is, in particular, a topological \( \mathbb{A} \)-algebra. Moreover, suppose that \( M_C(E,G), M_A(F,G) \) are locally equicontinuous, where \( G \) is a unital complete topological \( \mathbb{A} \)-algebra with continuous multiplication and let \( \tau \) be a compatible topology on \( E \otimes F \) satisfying (2.2), (2.4). If \( Q \) is the set (2.7), then

\[
M_A(E \hat{\otimes}_\tau F, G) = Q = M_A(E \otimes F, G) \subseteq M_C(E, G) \times M_A(F, G)
\]

(2.14)

within homeomorphisms. In case \( G \) is commutative, the “inclusion map” in (2.14) may be replaced by an equality.

Proof: Lemma 2.1 shows that \( M_A(E \otimes F, G) \) is locally equicontinuous so that \( M_A(E \hat{\otimes}_\tau F, G) = M_A(E \otimes F, G) \) (cf. (2.13)). Thus, the assertion is immediate from Proposition 2.1.

If \( E \) is a topological \( \mathbb{A} \)-algebra, a non-zero continuous \( \mathbb{A} \)-morphism \( f: E \to \mathbb{A} \) is called a continuous \( \mathbb{A} \)-character of \( E \).

Now, the set \( M_A(\mathbb{A}, \mathbb{A}) \) is equal to \( \{\text{id}_\mathbb{A}\} \) (identity of \( \mathbb{A} \)) since for any \( f \in M_A(\mathbb{A}, \mathbb{A}), a \in \mathbb{A} \), one has \( f(a) = f(a \cdot 1_\mathbb{A}) = f(a \cdot 1_\mathbb{A}) = a \cdot f(1_\mathbb{A}) = a \cdot 1_\mathbb{A} = a \).

Thus, by Proposition 2.1 there exists a map

\[
M_A(E \otimes_\tau \mathbb{A}, \mathbb{A}) \to M_C(E, \mathbb{A}) \times \{\text{id}_\mathbb{A}\}: h \mapsto (f, \text{id}_\mathbb{A}),
\]

such that

\[
h = f \otimes \text{id}_\mathbb{A}.
\]

(2.15)

So, Proposition 2.1 and Theorem 2.1 yield the next

Corollary 2.1. Let \( E, \mathbb{A} \) be unital topological algebras with continuous multiplications and \( \tau \) a compatible topology on \( E \otimes \mathbb{A} \) satisfying (2.2), (2.3). Moreover, let \( S \) be the subset of \( M_C(E, \mathbb{A}) \) consisting of all \( f \in M_C(E, \mathbb{A}) \) such that \( \mathbb{A} = (\text{Im} f)' := \{a \in \mathbb{A}: a \cdot f(x) = f(x) \cdot a, x \in E\} \). Then

\[
M_A(E \otimes_\tau \mathbb{A}, \mathbb{A}) = S,
\]

(2.17)

within a homeomorphism. Furthermore, if \( \mathbb{A} \) is complete, \( M_A(E, \mathbb{A}) \) is locally equicontinuous and \( \tau \) satisfies (2.4), then

\[
M_A(E \hat{\otimes}_\tau \mathbb{A}, \mathbb{A}) = M_A(E \otimes_\tau \mathbb{A}, \mathbb{A}) = S.
\]

(2.18)
In case $A$ is commutative, the relations (2.17), (2.18) take the form

\[(2.19) \quad M_A(E \otimes A, A) = M_C(E, A), \quad M_A(E \hat{\otimes} A, A) = M_C(\hat{E}, A), \]

respectively. □

Corollary 2.1 and relation (2.5) gives the following.

**Corollary 2.2.** Let $X$ be a completely regular $k$-space and $A$ a complete locally convex algebra with continuous multiplication. Then each $h \in M_A(C_c(X, A), A)$ is of the form

\[(2.20) \quad h = f \otimes \text{id}_A ,\]

with $f \in M_C(C_c(X), A)$. In particular, if $A$ is commutative, then

\[(2.21) \quad M_A(C_c(X, A), A) = M_C(C_c(X), A)\]

within a homeomorphism. □

One has analogous results to Corollary 2.2 in the case of vector-valued holomorphic (resp. $C^\infty$-) functions (cf. (2.6) and also §4 below).

Under suitable conditions analogous to [2: §3] the results of this section are also in force if the involved algebras have bounded approximate identities instead of identities.

### 3 - C-central $A$-morphisms

In this section we examine a new class of central morphisms, as the title indicates which is different from that of central morphisms defined in [2,3].

So a continuous $A$-morphism $h: E \to F$ between two topological $A$-algebras $E$, $F$ with identities $1_E$, $1_F$ is said to be C-central if the center of $\overline{\text{Im} h}$ (closure of $\text{Im} h$ in $F$) is $C$-trivial, in the sense that

\[(3.1) \quad G(\overline{\text{Im} h}) \equiv \overline{\text{Im} h} \cap (\overline{\text{Im} h})' = C \cdot 1_F \simeq C \subseteq A .\]

Denoting by $M^0_A(E, F)_C$ this set of morphisms, we endow it with the simple convergence topology in $E$, being thus a subset of $M_A(E, F) \subseteq L_A(E, F)_s$.

Each C-central $A$-morphism is a central $A$-morphism as in [3], while if, in addition, $\text{Im} h$ is closed, is as in [2]. The converse is true in case $A = C$. 

Now, if $E$ is a commutative topological $\mathbf{A}$-algebra, every $h \in M^0_A(E, F)_C$ takes the form
\begin{equation}
(3.2) \quad h = \chi \otimes 1_F ,
\end{equation}
where $\chi \in \mathcal{M}(E)$ (spectrum of $E$) such that $(\chi \otimes 1_F)(x) := \chi(x) \cdot 1_F$, $x \in E$. Indeed, by (3.1) $G(\text{Im} \, h) = \text{Im} \, h = C \cdot 1_F$ such that $f(x) = \lambda_x \cdot 1_F$, $\lambda_x \in C$, for every $x \in E$; hence one defines a map $\chi: E \to C: x \mapsto \chi(x) := \lambda_x$ which is an element of $\mathcal{M}(E)$ such that $f(x) = \chi(x) \cdot 1_F = (\chi \otimes 1_F)(x)$. Thus, since $C \simeq C \cdot 1_F \subseteq F$, one obtains the next homeomorphism
\begin{equation}
M^0_A(E, F)_C = \mathcal{M}(E) .
\end{equation}

The following theorem is analogous to [2: Propositions 4.1, 4.2] and [3: Lemma 1.1] in the present framework.

**Theorem 3.1.** Let $E$, $F$ be unital topological algebras with $E$ commutative and $F$ a topological $\mathbf{A}$-algebra. Moreover let $G$ be a complete unital topological $\mathbf{A}$-algebra with continuous multiplication and $\tau$ a compatible topology on $E \otimes F$ satisfying (2.2), (2.3). Then,
\begin{equation}
(3.4) \quad M^0_A(E \otimes \tau F, G)_C = \mathcal{M}(E) \times M^0_A(F, G)_C
\end{equation}
within a homeomorphism. Moreover, if $E$, $F$ have continuous multiplications, $\tau$ satisfies (2.4) and $\mathcal{M}(E)$, $M^0_A(F, G)_C$ are locally equicontinuous, then
\begin{equation}
(3.5) \quad M^0_A(E \otimes \tau F, G)_C = \mathcal{M}(\widehat{E}) \times M^0_A(\widehat{F}, G)_C
\end{equation}
within a homeomorphism.

**Proof:** Each $h \in M^0_A(E \otimes \tau F, G)_C$ is of the form $h = f \otimes g$ with $(f, g) \in M_C(E, G) \times M_A(F, G)$ (cf. Proposition 2.1) such that the commutativity of $E$ yields
\[ G(\text{Im} \, g) \subseteq G(\text{Im} \, h) , \quad G(\text{Im} \, f) = \text{Im} \, f \subseteq G(\text{Im} \, h) . \]
Thus, for $h$ as above one has $(f, g) \in \mathcal{M}(E) \times M^0_A(F, G)_C$ (cf. (3.6), (3.3)). Conversely, if $(f, g) \in M^0_C(E, G)_C \times M^0_A(F, G)_C$ then $f(x) = \lambda_x \cdot 1_G$, $\lambda_x \in C$ (cf. (3.3), (3.6)) such that
\[ f(x) \cdot g(y) = \lambda_x \cdot g(y) = g(y) \cdot f(x) \]
for all $(x, y) \in E \times F$, so that $h = f \otimes g \in M_A(E \otimes \tau F, G)$ (cf. Proposition 2.1). Moreover,
\[ C \cdot 1_G = G(\text{Im} \, f) \subseteq G(\text{Im} \, h) \subseteq G(\text{Im} \, g) = C \cdot 1_G , \]
i.e., \( h \in M^0_A(E \otimes F, G)_C \). That is, \( h = f \otimes g \) is a \( C \)-central \( A \)-morphism iff this is true for \( f, g \), so that (3.4) is immediate from Proposition 2.1, relation (3.3).

Now, one obtains that \( M^0_A(E \otimes F, G)_C \) is locally equicontinuous iff \( M(E), M^0_A(F, G)_C \) are locally equicontinuous (cf. also Lemma 2.1) such that \( M^0_A(E \otimes F, G)_C = M^0_A(E \otimes F, G)_C \) (cf. also (3.1), (3.6)); hence (3.4) implies (3.5).

Corollary 2.1 and Theorem 3.1 imply the following.

**Corollary 3.1.** Let \( E, A \) be unital complete topological algebras with continuous multiplications and \( E \) commutative. Moreover, let \( \tau \) be a compatible topology on \( E \otimes A \) satisfying (2.2), (2.3) and let \( M(E) \) be locally equicontinuous. Then one has

\[
M^0_A(E \otimes \tau A, A)_C = M(E)
\]

within a homeomorphism.

Corollary 3.1 and the relations (2.5), (2.6) (cf. also [6: Chapter VII, Theorems 1.2, 2.1]) prove the following corollaries.

**Corollary 3.2.** Let \( X \) be a locally compact space and \( A \) a unital complete locally convex algebra with continuous multiplication. Then,

\[
M^0_A(C_c(X, A), A)_C = X
\]

within a homeomorphism of the respective spaces.

**Corollary 3.3.** Let \( X \) be an \( n \)-dimensional compact \( C^\infty \)-manifold and \( A \) a unital Fréchet locally convex algebra. Then,

\[
M^0_A(C^\infty(X, A), A)_C = X
\]

within a homeomorphism of the respective spaces.

4 – Envelopes of holomorphy

We use in this section the preceding by studying the algebra of vector-valued holomorphic functions.
Let $X$ be a complex (analytic) manifold, $K$ a compact subset of $X$ and $E$ a unital complete locally convex $A$-algebra with continuous multiplication. Then, $O(K) \hat{\otimes} E$ is a locally convex $A$-algebra (cf. (2.6) and also [5: §2]).

Now, if $X$ is second countable, $(U_n)$ a (denumerable) fundamental system of open neighbourhoods of $K$ in $X$ and $E$ a complete locally $m$-convex $A$-algebra, for which the respective topological vector space is a $DF$-space, then

\[(4.1) \quad O(K, E) = O(K) \hat{\otimes} E\]

within an isomorphism of locally convex $A$-algebras (cf. [7] and relation (2.6)). Theorem 3.1 and relation (4.1) prove the following.

**Lemma 4.1.** Let $X$ be a complex (analytic) manifold which is second countable and $(U_n)$ a (denumerable) fundamental system of open neighbourhood of a compact subset $K$ of $X$. Moreover, let $E$ be a unital complete Fréchet locally $m$-convex $A$-algebra and $G$ a unital complete locally convex $A$-algebra with continuous multiplication such that $\mathcal{M}(O(K)), M_0^0(E, G)_C$ are locally equicontinuous. Then one has

\[(4.2) \quad M_0^0(\mathcal{O}(K, E), G)_C = \mathcal{M}(O(K)) \times M_0^0(E, G)_C\]

within a homeomorphism. ■

In this concern, let $K$ be a compact subset of a Stein manifold $X$ and $(U_n)$ an open basis of neighbourhoods of $K$ in $X$. Considering the respective sequence $(\tilde{U}_n)_{n \in \mathbb{N}}$ with $\tilde{U}_n = \mathcal{M}(O(U_n)), n \in \mathbb{N}$, one gets a decreasing sequence of Stein manifolds containing $K$, such that

\[(4.3) \quad \mathcal{M}(O(K)) = \varprojlim \mathcal{M}(O(U_n)) = \varprojlim \tilde{U}_n = \bigcap_n \tilde{U}_n = K\]

within homeomorphisms (cf. also [6: p. 163]). Thus, Lemma 4.1 and (4.3) yield the following homeomorphism

\[(4.4) \quad M_0^0(\mathcal{O}(K, E), G)_C = K \times M_0^0(E, G)_C .\]

In particular, if $E = A = G$, the last relation gives the following homeomorphism

\[(4.5) \quad M_0^0(\mathcal{O}(K, A), A) = K ;\]

i.e. the set of $C$-central $A$-morphisms of $O(K, A)$, with respect to $A$, is a compact subset of a Stein manifold (cf. also [4, 5]).
On the other hand, if \((X, O)\) is a complex space and \(A\) an open subset of \(X\), we shall say that \((X, A)\) is a Runge pair, with respect to the (structure) sheaf \(O\), if the topological algebra \(O(X) \subseteq O(A)\) is dense in \(O(A)\) (cf. [7]).

Let \((X, O)\) be a complex space with \(X\) second countable and \(K\) a subset of \(X\). Moreover, let \((U_n)\) be a (denumerable) fundamental system of open neighbourhoods of \(K\) in \(X\), such that \((X, U_n), n \in \mathbb{N}\), is a Runge pair with respect to \(O\). If \(E\) is a unital Fréchet locally convex algebra, then \((X, U_n)\) is a Runge pair with respect to \(O \otimes E\) (cf. [7: Corollary 3.1, p. 370]), such that \((X, K)\) is also a Runge pair with respect to \(O \otimes E\). So we now have the following.

**Theorem 4.1.** Let \((X, O)\) be a complex space with \(X\) second countable and \(K\) a subset of \(X\). Moreover, let \((U_n)\) be a (denumerable) fundamental system of open neighbourhoods of \(K\) in \(X\), such that \((X, U_n), n \in \mathbb{N}\), is a Runge pair with respect to \(O\). Furthermore, let \(E\) be a unital Fréchet locally convex \(A\)-algebra and \(G\) a unital complete locally convex \(A\)-algebra with continuous multiplication such that \(M^0_{A}(E, G)_C\) is locally equicontinuous. Then

\[
M^0_{A}(O(X, E), G)_C = M(O(X)) \times M^0_{A}(E, G)_C
\]

within a homeomorphism.

**Proof:** The hypotheses and the above comments (cf. also Theorem 3.1) yield

\[
M^0_{A}(O(X, E), G)_C = M^0_{A}(O(K, E), G)_C = M^0_{A}(O(K, E), G)_C
\]

within homeomorphisms. Thus, Lemma 4.1 and the preceding definitions show the homeomorphism (4.6).  

As an application of the above we have the following.

**Corollary 4.1.** Let the hypotheses of Theorem 4.1 be satisfied such that in particular \(E = G = A\). Then,

\[
\text{Env}_{O(X, A)}(X) = \text{Env}_{O(X)}(X)
\]

within a homeomorphism.

**Proof:** The hypotheses in connection with Theorem 4.1, Corollary 3.1 give the homeomorphism

\[
M^0_{A}(O(X, A), A)_C = M(O(X))
\]

In [5] (cf. also [4]) the first member of (4.8) was proved to be a Riemann domain and moreover an \(O(X, A)\)-convex set in the case \(X\) is a Riemann domain; hence,
it is, by definition, Env_{O,X,A}(X). Moreover, the second member of (4.8) is, by
definition the classical Env_{O,X}(X) (cf. [6: Chapter V, Definition 4.1]). Thus
(4.7) is an immediate consequence of (4.8) and the previous comments. ■

The above Corollary 4.1 lead us to the following.

**Theorem 4.2** (A. Mallios). Let the hypotheses of Corollary 4.1 be satisfied.
Then Env_{O,X,A}(X) is independent of the algebra A, range of the holomorphic
functions considered. ■

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