JACOBI ACTIONS OF $SO(2) \times \mathbb{R}^2$ AND $SU(2, \mathbb{C})$
ON TWO JACOBI MANIFOLDS

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Abstract: We take a sphere $S$ of the dual space $G^*$ of $G = so(2) \times \mathbb{R}^2$ with the Jacobi manifold structure obtained by quotient by the homothety group of the Lie-Poisson structure in $G^* \setminus \{0\}$ and we study the actions of two subgroups of $SO(2) \times \mathbb{R}^2$ on $S$.

We show that the natural action of $SU(2, \mathbb{C})$ on the unitary 3-sphere of $\mathbb{C}^2$ with the Jacobi structure determined by its canonical contact structure is a Jacobi action that admits an unique $\text{Ad}^*$-equivariant momentum mapping.

1 – Introduction

The notions of Jacobi manifold and Jacobi conformal manifold were introduced by A. Lichnerowicz ([5]) in 1978. A. Kirillov ([3]) also studied these structures under the name of local Lie algebras, when defined on the space of the differentiable sections of a vector bundle with 1-dimensional fibres.

Let $G^*$ be the dual of the Lie algebra of a finite dimensional Lie group, with its Lie–Poisson structure ([6]), and take the quotient of $G^* \setminus \{0\}$ by the homothety group. A. Lichnerowicz ([6]) showed that the Lie–Poisson structure defines on the quotient space (which can be identified with an unitary sphere of $G^*$) a Jacobi structure.

Finally, let us recall that the notion of momentum mapping, introduced by J.-M. Souriau ([11]) and B. Kostant ([4]) in the symplectic manifold context, can be extended to the Jacobi manifolds (cf. [8]), when a Jacobi action or a conformal Jacobi action ([9]) of a Lie group on a Jacobi manifold takes place.

In Appendix we summarize some of the basic concepts useful for a better understanding of the paper.

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2 – A Jacobi action of the Lie group $SO(2) \times \mathbb{R}^2$ on the unitary sphere of the dual of its Lie algebra

Let $G$ be the Lie group of the euclidean displacements, that is, the semidirect product of $SO(2)$ with $\mathbb{R}^2$. The product of two elements $(g, x)$ and $(h, y)$ in $G = SO(2) \times \mathbb{R}^2$ is given by

\[(g, x) \cdot (h, y) = (gh, gy + x) .\]

We can write the elements $(g, x)$ of $G$ as $3 \times 3$ matrices of the form

\[
\begin{pmatrix}
\cos \alpha & -\sin \alpha & x_1 \\
\sin \alpha & \cos \alpha & x_2 \\
0 & 0 & 1
\end{pmatrix} \equiv (g, x),
\]

where $\alpha \in \mathbb{R}$, $(x_1, x_2) \in \mathbb{R}^2$, the composition law (1) in $G$ corresponding to the product of the two respective matrices.

The Lie group $G$ acts on the plane $\mathbb{R}^2$ by an action $\phi$ given by

\[\phi: ((g, x), y) \in G \times \mathbb{R}^2 \rightarrow (g, y + x) \in \mathbb{R}^2\]

which can be expressed in matricial form by the following product of matrices:

\[
\begin{pmatrix}
\cos \alpha & -\sin \alpha & x_1 \\
\sin \alpha & \cos \alpha & x_2 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
1
\end{pmatrix} =
\begin{pmatrix}
y_1 \cos \alpha - y_2 \sin \alpha + x_1 \\
y_1 \sin \alpha + y_2 \cos \alpha + x_2 \\
1
\end{pmatrix} .
\]

This action corresponds to an $\alpha$-rotation of the point $(y_1, y_2)$ about the origin followed by a translation by the vector of components $(x_1, x_2)$.

Let $\mathcal{G} \equiv so(2) \times \mathbb{R}^2$ be the Lie algebra of $G$. An element $(a, v)$ of $\mathcal{G}$ can be written as

\[
\begin{pmatrix}
0 & a & v_1 \\
-a & 0 & v_2 \\
0 & 0 & 0
\end{pmatrix} \equiv (a, v),
\]

where $a \in \mathbb{R}$ and $(v_1, v_2) \in \mathbb{R}^2$.

The set $\mathcal{B}$ of elements

\[
B_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\]

is a basis of $\mathcal{G} = so(2) \times \mathbb{R}^2$. Let

\[
\left\{ \frac{\partial}{\partial B_1}, \frac{\partial}{\partial B_2}, \frac{\partial}{\partial B_3} \right\}
\]
be the basis of $G^*$, dual of $B$. Once we have

$$[B_1, B_2] = -B_3, \quad [B_1, B_3] = B_2 \quad \text{and} \quad [B_2, B_3] = 0,$$

if we put

$$\Lambda = -B_3 \frac{\partial}{\partial B_1} \wedge \frac{\partial}{\partial B_2} + B_2 \frac{\partial}{\partial B_1} \wedge \frac{\partial}{\partial B_3}$$

and

$$Z = \sum_{i=1}^{3} B_i \frac{\partial}{\partial B_i},$$

the couple $(\Lambda, Z)$ defines an homogeneous Lie–Poisson structure on $G^*$. (Homogeneous means that $[\Lambda, Z] = -\Lambda$, $[,]$ being the Schouten bracket ([10]); $Z$ is called the Liouville vector field.)

From now on, we will identify $G^* = (\text{so}(2) \times \mathbb{R}^2)^*$ with the product $(\text{so}(2))^* \times (\mathbb{R}^2)^*$. Thus, an arbitrary element of $G^*$ will be expressed by a couple $(\xi, p)$ with $\xi \in (\text{so}(2))^*$ and $p \in (\mathbb{R}^2)^*$.

Let us suppose that $G^*$ is endowed with the usual Euclidean norm. If $\eta = (\xi, p)$ is an element of $G^*$ with coordinates $(\eta_1, \eta_2, \eta_3)$ in the basis $\left\{ \frac{\partial}{\partial B_i} \right\}$, we define the norm of $\eta$, by putting

$$\|\eta\|^2 = \sum_{i=1}^{3} (\eta_i)^2.$$

Let $S$ be the unitary sphere of $G^*$,

$$S = \left\{ \eta \in G^*: \|\eta\|^2 = 1 \right\},$$

and suppose that $S$ is supplied with the Jacobi structure obtained by quotient of the Lie–Poisson structure of $G^*_0 = G^* \setminus \{0\}$ by the homothety group. On the open subsets

$$U_i^+ = \left\{ (B_1, B_2, B_3) \in S: B_i > 0 \right\}$$

and

$$U_i^- = \left\{ (B_1, B_2, B_3) \in S: B_i < 0 \right\}, \quad i = 1, 2, 3$$

of $S$, we take the coordinate functions

$$\left( x_1 = B_1, \quad \tilde{x}_i = \tilde{B}_i, \quad x_3 = B_3 \right), \quad i = 1, 2, 3,$$

where “-” means absence.

The Jacobi structure $(C, E)$ of $S$ is given, in the local charts taken above, in the following Table, where

$$\varepsilon = \begin{cases} +1, & \text{on } U_i^+, \\ -1, & \text{on } U_i^- . \end{cases}$$
V. Guillemin and S. Sternberg ([2]) showed that the coadjoint action $\text{Ad}^*$ of $G$ on the dual $G^*$ of its Lie algebra is given by

$$\text{Ad}^*_{(g_0, x)}(\xi, p) = \left(\xi + (g_0 p) \otimes x, g_0 p\right),$$

for every $(g_0, x) \in G$ and $(\xi, p) \in G^*$, where $\otimes$ is a mapping from $(\mathbb{R}^2)^* \times \mathbb{R}^2$ to $(\text{so}(2))^*$,

$$(p, x) \in (\mathbb{R}^2)^* \times \mathbb{R}^2 \to p \otimes x \in (\text{so}(2))^*,$$

such that

$$\langle p \otimes x, a \rangle = \langle p, ax \rangle,$$

for all $a \in \text{so}(2)$.

The restriction to $S$ of the coadjoint action of $G$ on $G^*$ doesn’t preserve the sphere $S$. However, we can take the quotient coadjoint action ([6]) $\overline{\text{Ad}}$ of $G$ on $S$ which is given, for every $(g_0, x) \in G$, by

$$\pi \circ \text{Ad}^*_{(g_0, x)} = \overline{\text{Ad}}(g_0, x) \circ \pi,$$

where $\pi : G_0^* \to S$ is the canonical projection of $G_0^*$ on the sphere $S$, this one being identified with the quotient of $G_0^*$ by the homothety group.

Let

$$H = \left\{ (g_0, 0), \ g_0 \in \text{SO}(2) \right\}$$
be the 1-dimensional Lie subgroup of $G$ corresponding to the plane rotations about the origin and whose elements can be written on the form

$$
\begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix} \equiv (g_\alpha, 0), \quad \alpha \in \mathbb{R}.
$$

From (2), we may conclude that the restriction $\text{Ad}^{*H}$ to the Lie subgroup $H$, of the coadjoint action of $G$ on $G^*$ is given by

$$
\text{Ad}^{*H}_{(g_\alpha, 0)}(\xi, p) = (\xi, g_\alpha p),
$$

with $(\xi, p) \in (\text{so}(2))^* \times (\mathbb{R}^2)^*$ and $(g_\alpha, 0) \in H$.

As the $\text{Ad}^{*H}$ action preserves the sphere $S$ (in fact if $(\xi, p) \in S$ then $(\xi, g_\alpha p) \in S$, since $\| (\xi, p) \| = \| (\xi, g_\alpha p) \|$), the restriction to the subgroup $H$ of the quotient action $\overline{\text{Ad}}$ of $G$ on $S$, coincides with the restriction to $S$ of the $\text{Ad}^{*H}$ action,

$$
\text{Ad}^{*H} = \overline{\text{Ad}}|_H : H \times S \to S.
$$

**Proposition.** The restriction to the subgroup $H$ of the quotient coadjoint action of $G$ on $S$ is a Jacobi action.

**Proof:** The Lie algebra of $H$ being generated by the element

$$
B_1 = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

of the basis $\mathcal{B}$ of $G$, the $\text{Ad}^{*H}$ action is a Jacobi action of $H$ on $S$ if

$$
[(B_1)_S, E] = 0 \quad \text{and} \quad [(B_1)_S, C] = 0,
$$

where $(B_1)_S$ is the fundamental vector field associated with $B_1$ ([9]) and in the last equality $[\ , \ ]$ is the Schouten bracket ([10]). But, if $X_{x_1}$ is the hamiltonian vector field ([7]) associated with $x_1 \in C^\infty(S, \mathbb{R})$, we have

$$(B_1)_S = X_{x_1},$$

because $B_1$, as a function from $G^*$ to $\mathbb{R}$, is homogeneous with respect to the Liouville vector field and projects into $S$, its projection being the function $x_1$. We have then

$$
[(B_1)_S, E] = [X_{x_1}, E] = X_{-(E, x_1)}
$$

and
$$[(B_1)_S, C] = [X_{x_1}, C] = -(E.x_1) C.$$ 

If we look at the expression of the vector field $E$ in the local charts of $S$ on the preceding Table, we can see that

$$E.x_1 = 0,$$

in all cases. Thus, we have

$$[(B_1)_S, E] = [(B_1)_S, C] = 0$$

and the $\text{Ad}^*H \equiv \overline{\text{Ad}}|_H$ action is a Jacobi action of $H$ on $S$. 

If instead of $H$ we take the 2-dimensional subgroup $H^1$ of $G$ that corresponds to the plane translations and whose elements are of the form

$$\begin{pmatrix}
1 & 0 & x_1 \\
0 & 1 & x_2 \\
0 & 0 & 1
\end{pmatrix},$$

where $(x_1, x_2) \in \mathbb{R}^2$, the restriction to $H^1$ of the quotient coadjoint action of $G$ on the sphere $S$ is a conformal Jacobi action. In fact, the Lie algebra of $H^1$ being generated by the elements

$$B_2 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \quad \text{and} \quad B_3 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}$$

of the basis $\mathcal{B}$ of $\mathcal{G}$, we have

$$\begin{cases}
[(B_2)_S, E] = [X_{x_2}, E] = X_{-(E.x_2)} \\
[(B_2)_S, C] = [X_{x_2}, C] = -(E.x_2)
\end{cases}$$

and also

$$\begin{cases}
[(B_3)_S, E] = [X_{x_3}, E] = X_{-(E.x_3)} \\
[(B_3)_S, C] = [X_{x_3}, C] = -(E.x_3)
\end{cases}.$$ 

Thus, the action $\overline{\text{Ad}}|_{H^1}$ is a conformal Jacobi action of $H^1$ on the Jacobi manifold $S$. 

3 – A Jacobi action of $SU(2,\mathbb{C})$ on the unitary 3-sphere of $\mathbb{C}^2$

Let $(z_1, z_2)$ be the canonical coordinates on $\mathbb{C}^2$. We take $\mathbb{C}^2$ with the following hermitian product

$$\left( (z_1, z_2) \left| (z'_1, z'_2) \right. \right) = z_1 \overline{z}'_1 + z_2 \overline{z}'_2 .$$

By means of this hermitian product, we can define a norm in $\mathbb{C}^2$ by putting

$$\| (z_1, z_2) \|^2 = \left( (z_1, z_2) \left| (z_1, z_2) \right. \right) = z_1 \overline{z}_1 + z_2 \overline{z}_2 .$$

Let

$$S^3 = \left\{ (z_1, z_2) \in \mathbb{C}^2 : z_1 \overline{z}_1 + z_2 \overline{z}_2 = 1 \right\}$$

be the unitary sphere of $\mathbb{C}^2$ and let $\alpha$ be the 1-form in $\mathbb{C}^2$ given by

$$\alpha = \text{Re} \left[ \frac{1}{i} (z_1 \, d\overline{z}_1 + z_2 \, d\overline{z}_2) \right] .$$

The restriction of $\alpha$ to $S^3$ defines a contact structure on the sphere ([11]).

If we identify the space $\mathbb{C}^2$ with $\mathbb{R}^4$, making the correspondence between the couple of complexes $(z_1 = x_1 + ix_3, z_2 = x_2 + ix_4)$ and the real quadruple $(x_1, x_2, x_3, x_4)$, the 1-form $\alpha$ express as

$$\alpha = -x_3 \, dx_1 - x_4 \, dx_2 + x_1 \, dx_3 + x_2 \, dx_4 .$$

Since every contact manifold is a Jacobi manifold ([5]), we can take the sphere $S^3$ as a Jacobi manifold whose structure is given by

$$E = -x_3 \frac{\partial}{\partial x_1} - x_4 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_4} ,$$

$$C = \frac{1}{2} (x_1 x_4 - x_2 x_3) \left( \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} \right)$$

$$- \frac{1}{2} (x_1 x_2 + x_3 x_4) \left( \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \right)$$

$$- \frac{1}{2} \left( (x_1)^2 + (x_3)^2 - 1 \right) \left( \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} \right)$$

$$- \frac{1}{2} \left( (x_2)^2 + (x_4)^2 - 1 \right) \left( \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_4} \right) .$$

Let’s take the Lie group $SU(2, \mathbb{C})$ — which is a Lie subgroup of $GL(2, \mathbb{C})$ of dimension (real) 3 — and its Lie algebra $su(2, \mathbb{C})$. According to its definition, $SU(2, \mathbb{C})$ preserves the norm in $\mathbb{C}^2$ and acts on $S^3$ by the natural action

$$\left( A, (z_1, z_2) \right) \in SU(2, \mathbb{C}) \times S^3 \to A \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) \in S^3 .$$
The elements 

$$X_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad X_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

that verify 

$$[X_1, X_2] = -2X_3, \quad [X_1, X_3] = -2X_2 \quad \text{and} \quad [X_2, X_3] = -2X_1,$$

set up a basis of $\text{su}(2, \mathbb{C})$. Taking into account the preceding identification of $\mathbb{C}^2$ with $\mathbb{R}^4$, we can write these elements on the following form:

\begin{align}
X_1 &= -x_4 \frac{\partial}{\partial x_1} - x_3 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4}, \\
X_2 &= x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4}, \\
X_3 &= x_3 \frac{\partial}{\partial x_1} - x_4 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_4}.
\end{align}

**Proposition.** The natural action of $\text{SU}(2, \mathbb{C})$ on the sphere $(S^3, C, E)$ is a Jacobi action.

**Proof:** The set $\{X_1, X_2, X_3\}$ being a basis of $\text{su}(2, \mathbb{C})$, we only must show that 

$$[(X_i)_{S^3}, E] = [(X_i)_{S^3}, C] = 0, \quad \text{for } i = 1, 2, 3,$$

where $(X_i)_{S^3}$ is the fundamental vector field associated with $X_i$, with respect to the action of $\text{SU}(2, \mathbb{C})$ on $S^3$. But, this action being the natural action, we have, for $i = 1, 2, 3$,

$$(X_i)_{S^3} = -X_i.$$

From (3) and (4), we can easily prove that

$$[X_i, E] = [X_i, C] = 0, \quad i = 1, 2, 3.$$

The action of $\text{SU}(2, \mathbb{C})$ on $S^3$ admits a momentum mapping that we’re going to evaluate. Let $A$ be an arbitrary element of $\text{SU}(2, \mathbb{C})$. Then $A$ is a matrix of the form

$$A = \begin{pmatrix} a + ib & c + id \\ -c + id & a - ib \end{pmatrix},$$

where $(a, b, c, d) \in \mathbb{R}^4$ and $a^2 + b^2 + c^2 + d^2 = 1$. 


Let $\xi$ be an element of $\text{su}^*(2, \mathbb{C})$ of coordinates $(\xi_1, \xi_2, \xi_3)$ on the dual basis of $\{X_1, X_2, X_3\}$. Then, for every $X \in \text{su}(2, \mathbb{C})$, we have

$$\langle \text{Ad}^*_A \xi, X \rangle = \langle \xi, \text{Ad}_{A^{-1}}(X) \rangle = \langle \xi, A^{-1}X A \rangle = \langle \xi, (A^T)^{-1} X A \rangle .$$

We also have, for the elements $X_1$, $X_2$ and $X_3$ of the $\text{su}(2, \mathbb{C})$ basis,

$$\begin{align*}
\langle \text{Ad}^*_A \xi, X_1 \rangle &= \langle \xi, (a^2 - b^2 - c^2 + d^2) X_1 + 2(ab + cd) X_2 + 2(ac - bd) X_3 \rangle, \\
\langle \text{Ad}^*_A \xi, X_2 \rangle &= \langle \xi, 2(cd - ab) X_1 + (a^2 - b^2 + c^2 - d^2) X_2 + 2(-ad - bc) X_3 \rangle, \\
\langle \text{Ad}^*_A \xi, X_3 \rangle &= \langle \xi, 2(-ac - bd) X_1 + 2(ad - bc) X_2 + (a^2 + b^2 - c^2 - d^2) X_3 \rangle .
\end{align*}$$

So,

$$\text{Ad}^*_A \xi = \begin{pmatrix}
\xi_1(a^2 - b^2 - c^2 + d^2) + 2\xi_2(ab + cd) + 2\xi_3(ac - bd) \\
2\xi_1(cd - ab) + \xi_2(a^2 - b^2 + c^2 - d^2) + 2\xi_3(-ad - bc) \\
2\xi_1(-ac - bd) + 2\xi_2(ad - bc) + \xi_3(a^2 + b^2 - c^2 - d^2)
\end{pmatrix} .$$

**Proposition.** Let $J: S^3 \to \text{su}^*(2, \mathbb{C})$ be the mapping given by

$$\begin{align*}
\langle J, X_1 \rangle (x_1 + ix_3, x_2 + ix_4) &= 2(-x_1 x_2 - x_3 x_4), \\
\langle J, X_2 \rangle (x_1 + ix_3, x_2 + ix_4) &= 2(-x_1 x_4 + x_2 x_3), \\
\langle J, X_3 \rangle (x_1 + ix_3, x_2 + ix_4) &= (x_1)^2 - (x_2)^2 + (x_3)^2 - (x_4)^2 ,
\end{align*}$$

where $X_1$, $X_2$ and $X_3$ are the elements of the $\text{su}(2, \mathbb{C})$ basis defined above. Then $J$ is the unique $\text{Ad}^*$-equivariant momentum mapping of the natural Jacobi action of $\text{SU}(2, \mathbb{C})$ on $S^3$.

**Proof:** If we calculate the hamiltonian vector fields $X_{\langle J, X_i \rangle} (i = 1, 2, 3)$ corresponding to the functions $\langle J, X_i \rangle$, we obtain

$$X_{\langle J, X_i \rangle} = -X_i .$$

But, as we have already remarked, $(X_i)_{S^3} = -X_i$. The mapping $J$ is then a moment mapping of the action of $\text{SU}(2, \mathbb{C})$ on $S^3$.

Let $A = \begin{pmatrix} a + ib & c + id \\ -c + id & a - ib \end{pmatrix} \in \text{SU}(2, \mathbb{C})$ and $z_1 = x_1 + ix_3$, $z_2 = x_2 + ix_4 \in S^3,$
be arbitrary elements. Then, we have
\[
J \left( A, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right) = J \left( \begin{pmatrix} (ax_1 - bx_3 + cx_2 - dx_4) + i(ax_3 + bx_1 + cx_4 + dx_2) \\ (-cx_1 - dx_3 + ax_2 + bx_4) + i(-cx_3 + dx_1 - bx_2 + ax_4) \end{pmatrix} \right)
\]
\[
= \begin{pmatrix} -2(ax_1 - bx_3 + cx_2 - dx_4)(-cx_1 - dx_3 + ax_2 + bx_4) - 2(ax_3 + bx_1 + cx_4 + dx_2)(-cx_3 + dx_1 - bx_2 + ax_4) \\ -2(ax_1 - bx_3 + cx_2 - dx_4)(-cx_3 + dx_1 - bx_2 + ax_4) + 2(-cx_1 - dx_3 + ax_2 + bx_4)(ax_3 + bx_1 + cx_4 + dx_2) \end{pmatrix}
\]
\[
= \text{Ad}^*_A \left( \begin{pmatrix} -2(x_1x_2 + x_3x_4) \\ -2(x_1x_4 - x_2x_3) \end{pmatrix} \right)
\]
\[
= \text{Ad}^*_A \left( J(x_1 + ix_3, x_2 + ix_4) \right).
\]

So, \( J \) is an \( \text{Ad}^* \)-equivariant momentum mapping.

Finally remark that, as \( \text{su}(2, \mathbb{C}) \) equals its derived algebra, if an \( \text{Ad}^* \)-equivariant momentum mapping exists, it is unique. \( \blacksquare \)

**APPENDIX**

In what follows, \( M \) is a differentiable connected finite dimensional manifold.

**i)** Let \( A \) (resp. \( B \)) be a \( p \)-times (resp. \( q \)-times) contravariant skew-symmetric tensor field on \( M \). The Schouten bracket ([10]) of \( A \) and \( B \) is a \((p + q - 1)\)-times contravariant skew-symmetric tensor field on \( M \), denoted by \([A, B]\), such that for any closed \((p + q - 1)\)-form \( \beta \),
\[
i([A, B]) \beta = (-1)^{(p+1)q} i(A) di(B) \beta + (-1)^p i(B) di(A) \beta,
\]
where \( i \) is the interior product.

Some of the properties of the Schouten bracket are:

- **i)** If \( p = 1 \), \([A, B] = \mathcal{L}(A)B \) is the Lie derivative of \( B \) with respect to \( A \);
- **ii)** \([A, B] = (-1)^{pq}[B, A] \);
- **iii)** If \( C \) is an \( r \)-contravariant skew-symmetric tensor field,
\[
S(-1)^{pq} [B, C], A] = 0.
\]
where $S$ means sum after circular permutation;

iv) $[A, B \wedge C] = [A, B] \wedge C + (-1)^{(p+1)q} B \wedge [A, C]$.

II) Let $C$ be a two times contravariant skew-symmetric tensor field on $M$ and $E$ a vector field on $M$. For any couple $(f, h)$ of functions on $M$, we set

$$\{f, h\} = C(df, dh) + f(E, h) - h(E, f)$$

and define a bilinear and skew-symmetric internal law on $C^\infty(M, \mathbb{R})$. This law satisfies the Jacobi identity (i.e., $S\{f, h, g\} = 0$) if and only if

$$[C, C] = 2E \wedge C \quad \text{and} \quad [E, C] = 0 \quad (5)$$

the bracket $[\, , \, ]$ being the Schouten bracket. In this case, we say that $\{\, , \, \}$ is a Jacobi bracket and $(M, C, E)$ is a Jacobi manifold. The space $C^\infty(M, \mathbb{R})$ with a Jacobi bracket is a local Lie algebra. If $E = 0$, the Jacobi manifold is a Poisson manifold.

If $(M, C, E)$ is a Jacobi manifold, there exists a vector bundle morphism

$$\#(\, ) : (TM)^* \to TM$$

that is given, for all $\alpha$ and $\beta$ in the same fiber of $(TM)^*$, by

$$\langle \beta, \#\alpha \rangle = C(\alpha, \beta) .$$

If $f \in C^\infty(M, \mathbb{R})$, we call $X_f = \#df + fE$ the Hamiltonian vector field associated with $f$ (7).

Let $(M, C, E)$ be a Jacobi manifold and $a \in C^\infty(M, \mathbb{R})$ a differentiable function that never vanishes. For all $f$ and $h$ elements of $C^\infty(M, \mathbb{R})$, we set

$$\{f, h\}^a = \frac{1}{a} \{af, ah\} .$$

The bracket $\{\, , \}^a$ is a Jacobi bracket and defines on $M$ a new Jacobi structure $(C^a, E^a)$, with

$$C^a = aC \quad \text{and} \quad E^a = \#da + aE .$$

We say that the structure $(C^a, E^a)$ is $a$-conformal to $(C, E)$. The equivalence class of all Jacobi structures on $M$, conformal to a given structure is called a conformal Jacobi structure on $M$.

Let $(M_1, C_1, E_1)$ and $(M_2, C_2, E_2)$ be two Jacobi manifolds. A differentiable mapping $\phi : M_1 \to M_2$ is called a Jacobi morphism if

$$\{f, h\}_{M_2} \circ \phi = \{f \circ \phi, h \circ \phi\}_{M_1} ,$$
for all $f, h \in C^\infty(M_2, \mathbb{R})$. We call $\phi$ an $a$-conformal Jacobi morphism if there exists a function $a \in C^\infty(M_1, \mathbb{R})$ that never vanishes, such that $\phi$ is a Jacobi morphism of $(M_1, C_1, E_1)$ into $(M_2, C_2, E_2)$.

A vector field $X$ on a Jacobi manifold $(M, C, E)$ is an in\textsuperscript{f}initesimal Jacobi automorphism (resp. in\textsuperscript{f}initesimal conformal Jacobi automorphism) if and only if $[X, C] = 0$ and $[X, E] = 0$ (resp. if and only if there exists a function $a \in C^\infty(M, \mathbb{R})$ such that $[X, C] = aC$ and $[X, E] = \#da + aE$).

**III** Let $(M, C, E)$ be a Jacobi manifold and $G$ a Lie group acting on the left on $M$, by an action $\Phi$. Suppose that for each $g \in G$ there exists a function $a_g \in C^\infty(M, \mathbb{R})$ that never vanishes and such that the mapping

$$\phi_g : x \in M \rightarrow \phi(g, x) \in M$$

is an $a_g$-conformal Jacobi morphism. Then the action $\phi$ is called a conformal Jacobi action. When, for all $g \in G$, the function $a_g \in C^\infty(M, \mathbb{R})$ is constant and equals 1, the action $\phi$ is called a Jacobi action. In this case, for any $g \in G$, the mapping $\phi_g$ is a Jacobi morphism.

Given an element $X$ of the Lie algebra $\mathcal{G}$ of $G$, the fundamental vector field associated with $X$ for the action $\phi$ ([9]), is the vector field $X_M$ on $M$, such that, for all $x \in M$,

$$X_M(x) = \frac{d}{dt}(\phi(\exp(-tX), x))_{t=0}.$$  

If $G$ is a connected Lie group, the action $\phi$ of $G$ on $M$ is a Jacobi action (resp. conformal Jacobi action) if and only if for all $X \in \mathcal{G}$, the fundamental vector field $X_M$ associated with $X$ is an in\textsuperscript{f}initesimal Jacobi automorphism (resp. in\textsuperscript{f}initesimal Jacobi conformal automorphism).

**IV** Let $G$ be a finite dimensional Lie group and $\mathcal{G}$ its Lie algebra. On the dual $\mathcal{G}^*$ of $\mathcal{G}$ we can define a Poisson structure, called the Lie–Poisson structure ([6]), by setting for all $f, h \in C^\infty(\mathcal{G}^*, \mathbb{R})$ and $\xi \in \mathcal{G}^*$,

$$\{f, h\}(\xi) = \langle \xi, \left[ df(\xi), dh(\xi) \right] \rangle,$$

with $[\, , \,]$ the Lie bracket on $\mathcal{G}$, $\langle \, , \, \rangle$ the duality product of $\mathcal{G}$ and $\mathcal{G}^*$ and where we identify the elements of $\mathcal{G}$ with linear mappings of $\mathcal{G}^*$ into $\mathbb{R}$.

If $Z$ is the Liouville vector field on $\mathcal{G}^*$ and $\Lambda$ is the Lie–Poisson tensor field on $\mathcal{G}^*$, one can show ([6]) that

$$[\Lambda, Z] = -\Lambda,$$

i.e., $(\mathcal{G}^*, \Lambda, Z)$ is an homogeneous Lie–Poisson structure.
REFERENCES


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