MAGNETIC VECTOR FIELDS: NEW EXAMPLES

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Abstract. In a previous paper, we introduced the notion of magnetic vector fields. More precisely, we consider a vector field \( \xi \) as a map from a Riemannian manifold into its tangent bundle endowed with the usual almost Kählerian structure and we find necessary and sufficient conditions for \( \xi \) to be a magnetic map with respect to \( \xi \) itself and the Kähler 2-form. In this paper we give new examples of magnetic vector fields.

1. Preliminaries

In [13] the authors define the notion of magnetic maps with the aim of generalizing the notion of magnetic trajectory on a Riemannian manifold. In fact, both magnetic curves and harmonic maps can be obtained as particular situations of magnetic maps.

Let \( f : N \to M \) be a smooth map between two Riemannian manifolds \( (N, h) \) of dimension \( n \) and \( (M, g) \) of dimension \( m \). Suppose that \( N \) is compact and let \( \xi \) be a global vector field on \( N \) having null divergence. Let \( \omega \) be a 1-form on \( M \). The energy of \( f \) is known as
\[
E(f) = \frac{1}{2} \int_N |df|^2 dv_h,
\]
where \( dv_h \) is the volume element on \( N \) and \( |df| \) is the Hilbert-Schmidt norm of the differential \( df \) given (in a point \( p \in N \)) by
\[
|df_p|^2 = \sum_{i=1}^n g_{f(p)}(f_*\partial_i, f_*\partial_i).
\]
Here \( \{e_i; i = 1, \ldots, n\} \) is an arbitrary orthonormal basis for \( T_pN \) and \( f_* : T_pN \to T_{f(p)}M \) is the tangent map of \( f \) at \( p \).

A smooth map \( f : (N, h) \to (M, g) \) which is a critical point of \( E(f) \) is called a harmonic map (see e.g., [11, 21]).

Let us now define the following functional for \( f \) associated to \( \xi \) and \( \omega \):
\[
\mathcal{P}(f) = \int_N \omega(df(\xi))dv_h.
\]
The Landau-Hall functional associated to \( \xi \) and \( \omega \) is defined by
\[
LH(f) = E(f) + \mathcal{P}(f).
\]

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Let $I$ be an open interval containing $0$. A smooth variation of $f$ is a smooth map $F : N \times I \to M$, such that $F(p, 0) = f(p)$. For the sake of simplicity we use the notation $f_\epsilon(p) = F(p, \epsilon)$. The variation vector field along $f$ is a section in the induced bundle $f^{-1}T(M)$ defined by $V(x) = \frac{\partial L}{\partial \epsilon} \big|_{\epsilon=0}(x)$.

**Definition 1.1.** The map $f$ is called magnetic with respect to $\xi$ and $\omega$ if it is a critical point of the Landau Hall integral $LH(f)$.

In what follows we compute the first variation $\frac{d}{d\epsilon}LH(f_\epsilon)\big|_{\epsilon=0}$. It is known from the theory of harmonic maps that

$$\frac{d}{d\epsilon}E(f_\epsilon)\big|_{\epsilon=0} = - \int_N g(\tau(f), V) \circ f \ dv_\nu,$$

where $\tau(f) := \text{trace}_\nu \nabla df$ is the tension field of $f$.

Let us focus on the integral $P$ and compute $\frac{d}{d\epsilon}P(f_\epsilon)\big|_{\epsilon=0}$. Consider local coordinates $x^1, \ldots, x^n$ on $N$ and $y^1, \ldots, y^m$ local coordinates on $M$. With respect to this setting, the map $f_\epsilon$ may be expressed as $y^\alpha = f_\alpha^\alpha(x)$, where $f_\alpha^\alpha$ are smooth functions on the domain of coordinates $x$ taking values in $\mathbb{R}$. From now on the indices $i, j, k$ range from $1$ to $n$, while the indices $\alpha, \beta, \gamma$ range from $1$ to $m$.

We have

$$P(f_\epsilon) = \int_N \omega_\alpha(f_\epsilon(x)) \frac{\partial f_\alpha^\gamma}{\partial x^\gamma}(x) \xi^\gamma(x) \ dv_\nu.$$

Compute

$$\frac{d}{d\epsilon} P(f_\epsilon) \big|_{\epsilon=0}$$

$$= \int_N \left[ \frac{\partial \omega_\alpha}{\partial y^\beta}(f(x)) \frac{\partial f_\alpha^\beta}{\partial \epsilon}(x) \bigg|_{\epsilon=0} + \omega_\alpha(f(x)) \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \frac{\partial f_\alpha^\gamma}{\partial x^\gamma}(x) \right] \xi^\gamma(x) \ dv_\nu.$$

Let us define a vector field $X$ on $N$ by $X(x) = \xi^\gamma(x)\omega_\alpha(f(x))V_\alpha^\gamma(x)\frac{\partial}{\partial x^\gamma}$ and compute its divergence. We obtain

$$\text{div}(X) = \left( \nabla_i \xi^\gamma \omega_\alpha(f(x))V_\alpha^\gamma(x) + \xi^\gamma(x)\nabla_\gamma \omega_\alpha(f(x))V_\alpha^\gamma(x) \right)$$

where $\nabla$ is the Levi-Civita connection on $N$ and $\nabla$ is the induced connection.

We successively have

$$\nabla_i \xi^\gamma = \text{div}(\xi),$$

$$\nabla_i \omega_\alpha(f(x)) = \left( \nabla_\gamma \omega_\alpha(f(x)) \right) \left( \frac{\partial}{\partial y^\gamma} \circ f \right) = \frac{\partial}{\partial x^\gamma} \omega_\alpha(f(x)) - \omega_\alpha(f(x)) \frac{\partial}{\partial x^\gamma}(f(x))\frac{\partial f_\gamma^\gamma(f(x))}{\partial x^\gamma} + \frac{\partial f_\gamma^\gamma(f(x))}{\partial x^\gamma} \omega_\alpha(f(x)).$$
\[ i\nabla_i V^\alpha = \frac{\partial V^\alpha}{\partial x^i}(x) + \frac{\partial f^\alpha}{\partial x^i}(x) \frac{\partial}{\partial x^i}(f(x))V^\alpha(x). \]

As \( \xi \) is divergence free, we get

\[ \text{div}(X) = \xi^i(x) \left[ \omega_\alpha(f(x)) \frac{\partial V^\alpha}{\partial x^i}(x) + \frac{\partial f^\alpha}{\partial x^i}(x) \frac{\partial \omega_\alpha}{\partial y^\alpha}(f(x))V^\alpha(x) \right]. \]

Since \( \int_N \text{div}(X)dv_h = 0 \), we obtain

\[ (1.2) \quad \int_N \xi^i(x)\omega_\alpha(f(x)) \frac{\partial V^\alpha}{\partial x^i}(x)dv_h = - \int_N \xi^i(x) \frac{\partial \omega_\alpha}{\partial y^\alpha}(f(x)) \frac{\partial f^\alpha}{\partial x^i}(x)V^\alpha(x)dv_h. \]

Combining (1.1) and (1.2) we find

\[ \frac{d}{de} \bigg|_{e=0} \mathcal{P}(f_e) = \int_N \xi^i(x) \frac{\partial \omega_\alpha}{\partial y^\alpha}(f(x)) \left( \frac{\partial f^\alpha}{\partial x^i}(x)V^\beta(x) - \frac{\partial f^\beta}{\partial x^i}(x)V^\alpha(x) \right)dv_h \]

\[ = \int_N \xi^i(x) \frac{\partial f^\alpha}{\partial x^i}(x) \left( \frac{\partial \omega_\alpha}{\partial y^\beta}(f(x)) - \frac{\partial \omega_\beta}{\partial y^\alpha}(f(x)) \right) V^\beta(x)dv_h \]

\[ = \int_N \xi^i(x) \frac{\partial f^\alpha}{\partial x^i}(x) (\omega_\alpha \omega_\beta V^\beta(x))dv_h \]

\[ = \int_N d\omega(f,\xi, V) \circ f dv_h. \]

Define the endomorphism \( \phi \), called the Lorentz force associated to the potential 1-form \( \omega \), by \( g(\phi(X), Y) = d\omega(X, Y) \), for all \( X, Y \) tangent to \( M \). It follows that

\[ \frac{d}{de} \bigg|_{e=0} \mathcal{P}(f_e) = \int_N g(\phi f, \xi, V) \circ f dv_h. \]

We finally obtain

\[ \frac{d}{de} \bigg|_{e=0} \mathcal{L}\mathcal{H}(f_e) = - \int_N g(\tau(f) - \phi f, \xi, V) \circ f dv_h. \]

We state the following.

**Theorem 1.1.** [13] Let \( f : (N, h) \to (M, g) \) be a smooth map. Then \( f \) is a magnetic map with respect to \( \xi \) and \( \omega \) if and only if it satisfies the Lorentz equation, that is

\[ (1.3) \quad \tau(f) = \phi(f, \xi). \]

Sometimes, equation [13] will be called the magnetic equation. Recall that on a Riemannian manifold \((M, g)\) a magnetic field is defined by a closed 2-form \( F \) and the Lorentz force associated to \( F \) is a \((1, 1)\) tensor field \( \phi \) on \( M \) given by \( g(\phi X, Y) = F(X, Y) \). The magnetic trajectories of \( F \) are curves \( \gamma \) satisfying the Lorentz equation \( \nabla_\gamma \gamma' = \phi \gamma' \). This equation is a particular case of equation (1.3) when \( N \) is an interval of \( \mathbb{R} \) and \( \xi = \frac{d}{dt} \), where \( t \) is the global coordinate on \( \mathbb{R} \). Magnetic curves were intensively studied in the last years by several geometers (including the authors of this article) in different ambient spaces. See for example [8, 9, 10, 11, 15, 18].
Remark 1.1. The Lorentz equation (1.3) was obtained from a variational principle assuming that the domain is compact and the 2-form $F$ is exact. Since it has a tensorial character, one can define a magnetic map $f : (N, h) \rightarrow (M, g)$ without the assumptions $N$ compact and $F$ exact (but only closed). Moreover, we will remove also the assumption for $\xi$ to be divergence free.

Let $\xi$ be a global vector field on $N$ and $F$ be a magnetic field on $M$ with the associated Lorentz force $\phi$. Similarly to magnetic curves, we may also introduce a strength (i.e., a real number) in the equation. Hence, we give the following.

Definition 1.2. We say that $f$ is a magnetic map with strength $q \in \mathbb{R}$ associated to $\xi$ and $F$ if the Lorentz equation
$$\tau(f) = q \phi(f, \xi)$$
is satisfied.

2. Vector fields as magnetic maps

In our previous paper [14], we ask when a vector field is a magnetic map. More precisely, we consider a Riemannian manifold $(M, g)$ of dimension $n$ and its tangent bundle $(T(M), g_S)$ equipped with the Sasaki metric. On $T(M)$ we also define an almost complex structure $J_S$ by
$$J_S X^H = X^V, \quad J_S X^V = -X^H,$$for all $X \in \mathfrak{X}(M)$.
It is known that $(T(M), g_S, J_S)$ is an almost Kählerian manifold [6]. Hence, the Kähler 2-form $\Omega_S = g_S(J_S \cdot, \cdot)$ may be considered as a magnetic field on $T(M)$.

A vector field $\xi \in \mathfrak{X}(M)$ will be thought as a map from $(M, g)$ to $(T(M), g_S, J_S)$. In the book of Dragomir and Perrone [7], the authors write the following formula
$$\tau(\xi) = -\{(\text{trace} g R(\nabla \xi, \xi) \xi^H + (\Delta_g \xi)^V) \circ \xi \}.$$Here $\Delta_g$ denotes the rough Laplacian on vector fields, defined by
$$\Delta_g X = -\sum_{k=1}^{n} [\nabla_{e_k} \nabla_{e_k} X - \nabla_{\nabla_{e_k} e_k} X],$$where $\{e_k\}_{k=1,...,n}$ is an orthonormal frame on $M$. We also have
$$J_S(\xi, \xi) = \xi^V - (\nabla \xi)^H.$$We state the following.

Theorem 2.1. [14] Let $(M, g)$ be a Riemannian manifold and $(T(M), g_S, J_S)$ its tangent bundle endowed with the usual almost Kählerian structure. Let $\xi$ be a vector field on $M$. Then $\xi$ is a magnetic map with strength $q$ associated to $\xi$ itself and the Kähler magnetic field $\Omega_S$ if and only if the following conditions hold:

(2.1) $\text{trace}_g R(\nabla \xi, \xi) \xi = q \nabla \xi \xi$,
(2.2) $\Delta_g \xi = -q \xi$.

Consider a Killing vector field $\xi$ on the Riemannian manifold $(M, g)$. We know that:
**Lemma 2.1.** A Killing vector field $\xi$ on a Riemannian manifold $(M,g)$ satisfies the equation $\nabla^2_{XY}\xi = -R(\xi,X)Y$, for all $X,Y \in \mathfrak{X}(M)$.

We ask now for $\xi : (M,g) \to (T(M),g_S,J_S)$ to be a magnetic map. Then $\xi$ must satisfy (2.2). But $\Delta g\xi = -\text{trace}_g \nabla^2\xi$. Using the previous lemma, we get $\Delta g\xi = \text{trace}_g R(\xi,\bullet)\bullet$.

On the other hand, we have

$$\text{Ric}(\xi,X) = \text{trace}_g \{Z \mapsto R_{Z\xi}X\} = \sum_{i=1}^n g(e_i, R_{e_i\xi}X) = -\sum_{i=1}^n g(R_{e_i\xi}e_i, X)$$

$$= g(\text{trace}_g R(\xi,\bullet)\bullet, X) = -qg(\xi, X), \text{ for all } X \in \mathfrak{X}(M).$$

So, if $Q$ is the Ricci operator, that is $g(Q\xi,\gamma) = \text{Ric}(\xi,\gamma)$, for all $\xi,\gamma$ tangent to $M$, then we get that $Q\xi = -q\xi$. We give the following.

**Proposition 2.1.** If a Killing vector field is a magnetic map with strength $q$, then it is an eigenvector of the Ricci operator corresponding to the eigenfunction $(-q)$.

**Remark 2.1.** In the special case of Einstein manifolds, the strength $q$ is related to the scalar curvature, namely $q = -\frac{\text{scal}}{n}$.

Suppose that $M$ is a real space form $M^n(c)$, case when the curvature tensor is expressed as $R_{XYZ} = c(g(Y,Z)X - g(X,Z)Y)$, for all $X,Y,Z \in \mathfrak{X}(M)$. We can easily compute $\text{trace}_g R(\nabla\xi,\xi)\bullet = c(\nabla\xi - \text{div}(\xi))$. As $\xi$ is Killing, its divergence is zero and thus, the magnetic equation becomes

$$(c - q)\nabla\xi \xi = 0.$$

We obtained the following.

**Theorem 2.2.** Let $\xi$ be a Killing vector field on a real space form $M^n(c)$, $n \geq 2$. If $\xi$ is a non-harmonic magnetic map with strength $q$, then $q = (1-n)c$ and $\xi$ is self parallel, case in which it has constant length.

**Proof.** Note that a real space form $M^n(c)$ is Einstein and its scalar curvature is $\text{scal} = cn(n-1)$. So, as $\xi$ is magnetic, cf. Remark 2.1 we must have $q = (1-n)c$. Obviously, equation (2.3) is satisfied if $q = c$. In this situation we get that $M$ is flat and $q = 0$, that is $\xi$ is a harmonic vector field. If $q \neq c$ then $\nabla\xi \xi = 0$. As $\xi$ is Killing, we have

$$g(\nabla\xi X, \xi) + g(\xi, \nabla_X \xi) = 0, \text{ for all } X \in \mathfrak{X}(M).$$

It follows that the length of $\xi$ is constant.

In the end of this section we propose the study of the following problem: **Study non-harmonic magnetic Killing vector fields on the unit sphere $S^n$.**
3. Magnetic vector fields on almost contact metric manifolds

A \((\varphi, \xi, \eta)\)-structure on a manifold \(M\) is defined by a field \(\varphi\) of endomorphisms of tangent spaces, a vector field \(\xi\) and a 1-form \(\eta\) satisfying

\[
\eta(\xi) = 1, \quad \varphi^2 = -1 + \eta \otimes \xi, \quad \varphi \xi = 0, \quad \eta \circ \varphi = 0.
\]

If \((M, \varphi, \xi, \eta)\) admits a compatible Riemannian metric \(g\), namely

\[g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \text{for all} \ X, Y \in \mathfrak{X}(M),\]

then \(M\) is said to have an almost contact metric structure, and \((M, \varphi, \xi, \eta, g)\) is called an almost contact metric manifold. It follows that \(\eta(X) = g(\xi, X)\), for any \(X \in \mathfrak{X}(M)\) and \(\xi\) is unitary.

The fundamental 2-form \(\Omega\) is defined by \(\Omega(X, Y) = g(\varphi X, Y)\), for any vector fields \(X\) and \(Y\). Recall that a contact metric manifold is an almost contact metric manifold such that \(\Omega = d\eta\). If in addition the structure is normal, that is the normality tensor field \(N = [\varphi, \varphi] + 2d\eta \otimes \xi\) vanishes, then the manifold \(M\) is called a Sasakian manifold. Here \([\varphi, \varphi]\) denotes the Nijenhuis tensor of \(\varphi\). Denoting by \(\nabla\) the Levi-Civita connection associated to \(g\), the Sasakian manifold \((M, \varphi, \xi, \eta, g)\) is characterized by \(\nabla X \varphi = -g(X, \varphi Y) \varphi X + \eta(Y)X\), for any \(X, Y \in \mathfrak{X}(M)\). As a consequence, we have \(\nabla X \xi = \varphi X\), for all \(X \in \mathfrak{X}(M)\). A systematic study of these structures is presented in the two books of Blair [4], [5]. However, we use the sign convention given by Sasaki, see e.g., [12].

On the other hand, a Kenmotsu manifold can be defined as a normal almost contact metric manifold such that \(d\eta = 0\) and \(d\Omega = 2\eta \wedge \Omega\). These manifolds can be characterized using their Levi-Civita connection, by requiring

\[\nabla_X \varphi = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \quad \text{for every} \ X, Y \in \mathfrak{X}(M).\]

In our previous paper [14], we find some conditions when the Reeb vector field \(\xi\) on a Sasakian space form is magnetic, that is satisfies the condition in Theorem 2.1. We obtain that \(q = -2n\).

Let us analyze the property of the characteristic vector field \(\xi\) on a Kenmotsu manifold to be magnetic. Recall the following two useful formulas:

\[\nabla_X \xi = X - \eta(X)\xi,\]

\[R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \quad \text{for every} \ X, Y \in \mathfrak{X}(M).\]

Compute \(\text{trace}_g R(\nabla_\xi, \xi)\). To do this, consider as usual, a \(\varphi\) adapted orthonormal basis \(\{e_i, \varphi e_i, \xi\}\), \(i = 1, \ldots, n\). We have \(\nabla_{e_i}\xi = e_i, \ \nabla_{\varphi e_i}\xi = \varphi e_i, \ \nabla_\xi = 0\). Hence

\[
\text{trace}_g R(\nabla_\xi, \xi) = \sum_{i=1}^{n} \left[ R(e_i, \xi) e_i + R(\varphi e_i, \xi) \varphi e_i \right] = \sum_{i=1}^{n} \left[ g(e_i, e_i) \xi + g(\varphi e_i, \varphi e_i) \xi \right] = 2n\xi.
\]

Thus, the equation (2.1) becomes \(2n\xi = 0\), which is a contradiction.
As a matter of fact, for the second condition of Theorem 2.1, we have
\[
\Delta_g \xi = - \sum_{i=1}^{n} \left[ (\nabla_{e_i} \nabla_{e_i} \xi - \nabla_{\nabla_{e_i} e_i} \xi) + (\nabla_{\varphi e_i} \nabla_{\varphi e_i} \xi - \nabla_{\nabla_{\varphi e_i} \varphi e_i} \xi) \right]
\]
\[
= - \sum_{i=1}^{n} \left[ \eta(\nabla_{e_i} e_i) \xi + \eta(\nabla_{\varphi e_i} \varphi e_i) \xi \right] = 2n \xi.
\]

Therefore, \( \xi \) is an eigenvector of the rough Laplacian with corresponding eigenfunction \( q = -2n \). We conclude with the following.

**Proposition 3.1.** The characteristic vector field of a Kenmotsu manifold is not magnetic.

Next we would like to make some comments on the same problem in a cosymplectic manifold. Recall that a cosymplectic manifold is an almost contact metric manifold for which the three tensor fields \( \varphi, \xi \) and \( \eta \) are parallel. Therefore, the first condition in the Theorem 2.1 is automatically satisfied. Since \( \Delta_g \xi = 0 \), the second condition implies \( q = 0 \), that is \( \xi \) is a harmonic map. We conclude with the following.

**Proposition 3.2.** If the characteristic vector field of a cosymplectic manifold is magnetic, then it is harmonic.

At this point we propose another problem:

**Study the property of \( \xi \) of being a magnetic map on a generalized Sasakian space form.** See [1].

We end this section with some comments concerning the condition \( \text{div}(\xi) = 0 \) used in finding the magnetic equation. Because some readers may think that the divergence free condition for \( \xi \) is too strong or artificial, we mention that this condition is often satisfied. For example, on almost contact metric manifolds, we know the following:

- The characteristic vector field \( \xi \) of a contact metric manifold is divergence free.
- In addition, cosymplectic manifolds have divergence free \( \xi \).
- However, \( \xi \) is not always divergence free; e.g. on Kenmotsu manifolds, we have

\[
\text{div} \ \xi = \sum_{i=1}^{n} g(\nabla_{e_i} \xi, e_i) + \sum_{i=1}^{n} g(\nabla_{\varphi e_i} \xi, \varphi e_i)
\]
\[
= \sum_{i=1}^{n} g(e_i, e_i) + \sum_{i=1}^{n} g(\varphi e_i, \varphi e_i) = 2n \neq 0.
\]

4. **More examples of magnetic maps**

4.1. **H-minimal submanifolds.** Let \( N \) be an \( n \)-dimensional Lagrangian submanifold in a Kähler manifold \( M \). Then \( \xi := -JH/n \) is a globally defined tangent vector field on \( M \). Here \( H \) is the mean curvature vector field. In our previous paper [13], we showed that the inclusion map \( \iota: N \to M \) satisfies \( \tau(\iota) = J_{\iota*} \zeta \).
According to Oh \cite{19}, a Lagrangian submanifold $N$ is said to be Hamiltonian-minimal (in short $H$-minimal) if it is a critical point of the volume functional under compactly supported smooth variations arising from Hamiltonian deformations.

The Euler–Lagrange equation of this variational problem is $\text{div} \, (JH) = 0$, that is $\zeta$ is divergence free.

This implies that every $H$-minimal Lagrangian submanifold $N$ is magnetic with respect to $\zeta = -JH/n$ and the Kähler form of $M$.

4.2. L-minimal submanifolds. In Sasakian geometry, one introduces the notion of $L$-minimal immersion as follows:

**Definition 4.1.** \cite{16} An $n$-dimensional Legendrian submanifold $N$ in a Sasakian manifold $M$ is said to be $L$-minimal if it is a critical point of the volume functional under compactly supported smooth variations arising from Legendre deformations.

The Euler–Lagrange equation of this variational problem is $\text{div}(\phi H) = 0$.

One can check that every Legendrian submanifold satisfies $\tau(\varphi) = \phi \chi$, where the vector field $\chi$ is defined globally on $N$ by $\chi := -\phi H/n$.

Thus every $L$-minimal Legendrian submanifold in a Sasakian manifold is magnetic with respect to the divergence free vector field $\chi$ and the contact form on $M$.

4.3. Magnetic hypersurfaces in complex space forms. Let $(M, g, J)$ be a Kähler manifold of complex dimension $n$ and let $f : N \to (M, g, J)$ be an orientable real hypersurface with unit normal vector field $\nu$. Then the Kähler structure $(g, J)$ induces an almost contact metric structure $(\phi, \chi, \eta, \varphi)$ on $N$ as follows. First, define the vector field $\chi$ by $f^* \chi = -J\nu$. Next $(\varphi, \eta)$ are defined by the formula $Jf^* X = f^*\varphi X + \eta(X)\nu$

for all tangent vector $X$ on $N$. Finally, we set $h = f^* g$.

Then the Levi-Civita connections $\nabla$ of $M$ and $\nabla$ of $N$ are related by the following **Gauss formula** and **Weingarten formula**:

$\tilde{\nabla}_X f_* Y = f_* \nabla_X Y + g(AX, Y)\nu$; \[\tilde{\nabla}_X \nu = -f_* AX, \quad X \in X(N).\]

The endomorphism field $A$ is called the **shape operator** of $N$ derived from $\nu$. We know that

$$(\nabla_X \varphi) Y = \eta(Y) AX - g(AX, Y)\chi, \quad \nabla_X \chi = \varphi AX.$$  

The following result is fundamental (see \cite{7}).

**Proposition 4.1.** The structure vector field $\xi$ is divergence free.

**Proof.** We can compute

$$\text{div} \, \xi = \sum_{i=1}^{n-1} g(\nabla e_i, \xi, e_i) + \sum_{i=1}^{n-1} g(\nabla \phi e_i, \phi e_i) + g(\nabla \xi, \xi)$$

$$= \sum_{i=1}^{n-1} g(\phi Ae_i, e_i) + \sum_{i=1}^{n-1} g(\phi A\phi e_i, \phi e_i) + g(\phi A\xi, \xi).$$
We note that $g(\phi A\xi, \xi) = 0$. Next we have

$$\langle \phi A\phi e_i, e_i \rangle = -\langle A\phi e_i, \phi^2 e_i \rangle = \langle A\phi e_i, e_i \rangle = \langle \phi e_i, Ae_i \rangle = -\langle e_i, \phi Ae_i \rangle.$$ 

Thus $\xi$ is divergence free.

The tension field $\tau(f)$ is given by $\tau(f) = (2n - 1)H\nu$. Here $H$ is the mean curvature function. If $\Omega = g(J, \cdot, \cdot)$ is considered as a magnetic field on $M$, then the magnetic equation for the immersion $f$ with respect to $\{\xi, \Omega\}$ and strength $q$ is computed as

$$(2n - 1)H\nu = qJ(f, \xi) = qJ(-J\nu) = q\nu.$$ 

Thus $f$ is magnetic with respect to $\{\xi, \Omega\}$ if and only if $q = (2n - 1)H$.

**Proposition 4.2.** [13] Let $f : N \to (M, g, J)$ be an orientable real hypersurface of constant mean curvature $H$ with induced almost contact metric structure $(\varphi, \xi, q, h)$. Then $f$ is a magnetic map with respect to the structure vector field $\xi$ and the Kähler magnetic field $\Omega$ with strength $q = (2n - 1)H$.

Now, we add one more example to our previous list of magnetic real hypersurfaces in complex space forms and complex Grassmannian manifolds given in [13], namely magnetic real hypersurfaces in complex quadrics.

**Example 4.1.** In [3], Berndt and Suh studied real hypersurfaces in the Grassmannian manifold $Gr_2(\mathbb{R}^{m+2})$ of oriented 2-planes in Euclidean $(m+2)$-space. As is well known, the Grassmannian manifold $Gr_2(\mathbb{R}^{m+2})$ is identified with the complex quadric

$$\Omega_m = \{[z_1 : z_2 : \cdots : z_{m+2}] \in \mathbb{C}P^{m+1} | z_1^2 + z_2^2 + \cdots + z_{m+2}^2 = 0\}$$

in the complex projective $(m + 1)$-space.

When we equip the ambient projective space with the Fubini–Study metric of constant holomorphic sectional curvature 4, then $\Omega_m = SO(m + 2)/SO(2) \times SO(m)$ is a Hermitian symmetric space of rank 2 and maximal sectional curvature 4 with respect to the induced metric $g$. The Ricci tensor is given by $\text{Ric} = 2mg$.

Hereafter we assume that $m \geq 3$. For $m = 2k$, the map

$$[z_1 : z_2 : \cdots : z_{k+1}] \mapsto [z_1 : z_2 : \cdots : z_{k+1} : iz_1 : iz_2 : \cdots : iz_{k+1}]$$

defines a totally geodesic complex immersion of $\mathbb{C}P^k$ into $\Omega_{2k} \subset \mathbb{C}P^{2k+1}$.

For $r \in (0, \pi/2)$, the tube around $\mathbb{C}P^k$ is a homogeneous real hypersurface with principal curvatures $\lambda_1 = 2\cot(2r)$, $\lambda_2 = 0$, $\lambda_3 = -\tan r$, $\lambda_4 = \cot r$ and multiplicities $m_1 = 1$, $m_2 = 2$, $m_3 = m_4 = 2k - 2$.

In case $m = 2$, i.e., $k = 1$, we have $\mathbb{C}P^1 \subset \Omega_2 = \mathbb{S}^2 \times \mathbb{S}^2$. The principal curvatures of a tube around $\mathbb{C}P^1$ are 0 and $2\cot(2r)$.

The inclusion map of a tube $M_r$ of radius $r$ around $\mathbb{C}P^k$ into $\Omega_{2k}$ is a magnetic immersion with respect to the magnetic field $F = \Omega$ with strength

$$q = m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 + m_4\lambda_4 = 2(2k - 1)\cot 2r.$$
4.4. Harmonic unit vector fields as magnetic maps. A unit vector field $\xi$ on a Riemannian manifold $(M,g)$ is said to be a harmonic unit vector field if it is a critical point of the energy functional over the space $X_1(M)$ of all smooth unit vector fields on $M$. The Euler-Lagrange equation of this variational problem is $\Delta_g \xi = |\xi|^2 \xi$. Moreover it is known that $\xi$ is a harmonic map from $(M,g)$ into the unit tangent sphere bundle $U(M)$ with the metric induced from $g_S$ if and only if $\xi$ is a harmonic unit vector field and satisfies
\[
\text{trace}_g R(\nabla_\xi \xi, \xi) = 0 \quad \text{(see [7]).}
\]
Comparing the harmonic map equation for $\xi : M \to U(M)$ and magnetic equation for $\xi : M \to T(M)$ we have

**Proposition 4.3.** Let $\xi$ be a unit vector field on a Riemannian manifold $(M,g)$. Assume that $\xi$ satisfies
- $\xi$ is divergence free, (optional condition)
- $\nabla_\xi \xi = 0$,
- $|\nabla_\xi| \text{ is constant}$
- $\xi : M \to U(M)$ is a harmonic map.
Then $\xi$ is a magnetic map into $T(M)$ with strength $q = -|\nabla_\xi|^2$.

4.5. Magnetic vector fields on real hypersurfaces. An oriented real hypersurface $N$ of a Kähler manifold $M$ is said to be Hopf if the structure vector field $\xi$ introduced in subsection 4.3 is a principal vector field. In that case, if $A\xi = \alpha \xi$, then $\alpha$ is called the Hopf principal curvature on $N$. It is easy to check that $\xi$ satisfies $\nabla_\xi \xi = 0$ if and only if $N$ is Hopf.

The following results are direct consequences of [20, Theorem 3.2] due to Perrone.

**Proposition 4.4.** Let $N \subset M$ be an oriented Hopf hypersurface of a Kähler–Einstein manifold. Then the structure vector field $\xi$ satisfies:

1. $\xi$ is a harmonic unit vector field if and only if $\text{grad } H = \xi(H) \xi$, where $H$ is the mean curvature function.
2. If the principal curvature $\alpha$ corresponding to $\xi$ is constant along the trajectories of $\xi$ then $\xi(H) = 0$.

**Corollary 4.1.** Let $N \subset M$ be an oriented Hopf hypersurface of a Kähler–Einstein manifold satisfying $\xi(\alpha) = 0$. Then $\xi$ is a harmonic map into $U(N)$ if and only if the mean curvature is constant.

Complex space forms are typical examples of Kähler–Einstein manifolds.

**Theorem 4.1.** Let $N$ be an oriented Hopf hypersurface with constant principal curvatures in a complex space form $M$. Then the characteristic vector field $\xi$ of $N$ is a magnetic map with strength $q = -|A|^2 + \alpha^2$.

**Proof.** Let $N$ be an oriented Hopf hypersurface with constant principal curvatures in a complex space form $M$. Then $\xi$ satisfies
\[
\Delta_g \xi = |\nabla_\xi|^2 \xi, \quad \text{trace}_g R(\nabla_\xi \xi, \xi) = 0, \quad \nabla_\xi \xi = 0.
\]
Since all the principal curvatures are constant and $\nabla_\xi = \varphi A$, we have $|\nabla_\xi|^2 = |A|^2 - \alpha^2$. Hence $\xi$ is a magnetic map with strength $q = -|A|^2 + \alpha^2$. \qed
As is well known, a complete and simply connected complex space form is a complex projective space $\mathbb{CP}^n(c)$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $\mathbb{CH}^n(c)$, according as $c > 0$, $c = 0$ or $c < 0$. Hopf hypersurfaces in $\mathbb{CP}^n(c)$ and $\mathbb{CH}^n(c)$ are classified by Kimura [17] and Berndt [2], respectively.

Of course, one can check that characteristic vector fields of all homogeneous Hopf real hypersurfaces in $\mathbb{CP}^n(c)$ and $\mathbb{CH}^n(c)$ are magnetic maps into tangent bundles. However, we exhibit here only few examples.

**Example 4.2 (Type A hypersurfaces).** Let us consider
$$\tilde{M}_k(r) := S^{2k+1}(\cos r) \times S^{2n-1-2k}(\sin r) \subset S^{2n+1},$$
$$0 \leq k < n, \quad 0 < r < \frac{\pi}{2}.$$ 

Then the Hopf projection image $M_k(r)$ of $\tilde{M}_k(r)$ is a Hopf hypersurface in the complex projective space $\mathbb{CP}^n(4)$ of constant holomorphic sectional curvature 4. These hypersurfaces $M_k(r)$ are referred as to type A hypersurfaces. Note that type A hypersurfaces are quasi-Sasakian. The type A hypersurface $M_k(r)$ has constant principal curvatures $\lambda_1 = -\tan r$, $\lambda_2 = \cot r$, $\alpha = 2 \cot(2r)$, $0 < r < \frac{\pi}{2}$ with multiplicities $m_1 = 2k$, $m_2 = 2(n - k - 1)$, $m_\alpha = 1$. Then the characteristic vector field $\xi$ is a magnetic map into $T(M)$ with strength
$$q = -(m_1\lambda_1^2 + m_2\lambda_2^2) = -2\{k\tan^2 r + (n - k - 1)\cot^2 r\} < 0.$$

**Example 4.3 (Horospheres).** Let $M$ be a horosphere in the complex hyperbolic $n$-space $\mathbb{CH}^n(-4)$. It is known that the horosphere in $\mathbb{CH}^n(-4)$ is a Sasakian space form of constant holomorphic sectional curvature $-3$. The horosphere has constant principal curvatures $\lambda = 1$ with multiplicity $2n - 2$ and $\alpha = 2$ with multiplicity 1. Then the strength is $q = -2(n - 1)$. This is consistent with Section 2.

**Example 4.4 (Type B hypersurfaces).** Let $M$ be a tube over totally real and totally geodesic real hyperbolic space $\mathbb{H}^n$ in the complex hyperbolic $n$-space $\mathbb{CH}^n(-4)$ of constant holomorphic sectional curvature $-4$. Then $M$ is a Hopf hypersurface with constant principal curvatures having the form
$$\lambda_1 = \frac{1}{r} \coth u, \quad \lambda_2 = \frac{1}{r} \tanh u, \quad \alpha = \frac{2}{r} \tanh (2u)$$

with multiplicities $m_1 = m_2 = n - 1$, $m_\alpha = 1$. Hence the characteristic vector field $\xi$ is a magnetic map into $T(M)$ with strength
$$q = -(m_1\lambda_1^2 + m_2\lambda_2^2) = -\frac{n - 1}{r^2}\{\coth^2 u + \tanh^2 u\} < 0.$$

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