TRANSFORMS FOR MINIMAL SURFACES
IN 5-DIMENSIONAL SPACE FORMS

Makoto Sakaki

Abstract. For a minimal surface in a 5-dimensional space form, we give transforms to get another minimal surface in another 5-or 4-dimensional space form.

1. Introduction

For a minimal surface in the 3-sphere $S^3$, the unit normal vector field, that is, the Gauss map gives another minimal surface in $S^3$ possibly with singularities (cf. [5]). It is generalized by Bolton, Pedit and Woodward [2] for superconformal minimal surfaces in odd-dimensional spheres. On the other hand, Bolton and Vrancken [3] discovered new transforms from a minimal surface with non-circular ellipse of curvature in the 5-sphere $S^5$, to another minimal surface in $S^5$, which are called $(\pm)$transforms (see also [1, 4]).

In this paper, generalizing them, we give transforms from a minimal surface in a 5-dimensional space form, to another minimal surface in another 5-or 4-dimensional space form.

Let $N^n(c)$ be the $n$-dimensional Riemannian space form of constant curvature $c$, where $c$ is either 1, 0 or $-1$. In particular, let $N^n(1) = S^n$, $N^n(0) = R^n$ and $N^n(-1) = H^n$. Let $R^{n+1}_i$ be the $(n+1)$-dimensional Minkowski space with standard coordinate system $(x_1, \cdots, x_n, x_{n+1})$ of signature $(+\cdots, +, -)$. Then

$$H^n = \{(x_1, \cdots, x_n, x_{n+1}) \in R^{n+1}_1 \mid x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1\},$$

and

$$S^n_i = \{(x_1, \cdots, x_n, x_{n+1}) \in R^{n+1}_i \mid x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = 1\},$$

where $S^n_i$ is the $n$-dimensional de Sitter space.

Let $f: M \rightarrow N^5(c)$ be an immersion of a 2-dimensional manifold $M$ into $N^5(c)$. We denote by $h$ the second fundamental form of $f$. The first normal space $T^+_1(x)$ at $x \in M$ is defined by

$$T^+_1(x) = \{h(X, Y) \mid X, Y \in T_xM\}.$$
The ellipse of curvature \( E(x) \) at \( x \in M \) is defined by
\[
E(x) = \{ h(X,X) \mid X \in T_x M, \ |X| = 1 \}.
\]

We assume that \( f : M \to N^5(c) \) is a minimal immersion. Suppose that the ellipse of curvature is non-degenerate at any point. Then the dimension of the first normal space is 2 at any point. Let \( e_5 \) be the unit normal vector to \( f(M) \) which is orthogonal to the first normal space. Then we can regard \( G = e_5 \) as a map to either \( S^5_5, S^4_4 \) or \( S^1_5 \), according to when \( c = 1, 0 \) or \( -1 \). It is the Gauss-like map.

**Theorem 1.1.** Let \( f : M \to N^5(c) \) be a minimal surface. Suppose that the ellipse of curvature is a non-degenerate circle at any point. If the Gauss-like map \( G \) is non-degenerate, then it gives a minimal surface in either \( S^5_5, S^4_4 \) or \( S^1_5 \).

**Remark 1.1.** The case \( c = 1 \) can be seen in [2].

Next we consider the case where the ellipse of curvature is not a circle. For a minimal surface \( f : M \to N^5(c) \), suppose that the ellipse of curvature is non-degenerate and non-circular at any point. Let \( a \) and \( b \) be the semi-minor and semi-major axes of the ellipse of curvature, respectively. We choose the local normal orthonormal frame field \( \{ e_\alpha \}_{3 \leq \alpha \leq 5} \) so that \( e_3 \) is in the direction of the semi-minor axis and \( e_4 \) is in the direction of the semi-major axis. Now, for \( \varepsilon = +1 \) or \( -1 \), let
\[
f^\varepsilon = \varepsilon \sqrt{1 - \left( \frac{a}{b} \right)^2 e_4 + \frac{a}{b} e_5}.
\]
Then \( f^\varepsilon \) is a map to either \( S^5_5, S^4_4 \) or \( S^1_5 \), according to when \( c = 1, 0 \) or \( -1 \).

**Theorem 1.2.** Let \( f : M \to N^5(c) \) be a minimal surface. Suppose that the ellipse of curvature is non-degenerate and non-circular at any point. Then \( f^\varepsilon \) gives a minimal surface in either \( S^5_5, S^4_4 \) or \( S^1_5 \).

**Remark 1.2.** It is a generalization of [3] for \( S^5_5 \).

## 2. Preliminaries

In this section, we recall the method of moving frames for surfaces in 5-dimensional space forms. We shall use the following convention on the ranges of indices:

\[
1 \leq A, B, \ldots \leq 5, \quad 1 \leq i, j, \ldots \leq 2, \quad 3 \leq \alpha, \beta, \ldots \leq 5.
\]

Let \( \{ e_A \} \) be a local orthonormal frame field in \( N^5(c) \), and \( \{ \omega^A \} \) be the dual coframe field. Let \( \omega^A_B \) denote the connection forms which satisfy \( \omega^A_B = -\omega^B_A \). The structure equations are given by
\[
d\omega^A = - \sum_B \omega^A_B \wedge \omega^B,
\]
(2.1)
\[
d\omega^A_B = - \sum_C \omega^A_C \wedge \omega^B_C + \frac{1}{2} \sum_{C,D} R^A_{BCD} \omega^C \wedge \omega^D, \quad R^A_{BCD} = c(\delta^A_C \delta_{BD} - \delta^A_D \delta_{BC}).
\]

Let \( f : M \to N^5(c) \) be a surface in \( N^5(c) \). When \( c = 1 \), \( f \) is an \( R^6 \)-valued map with \( \langle f, f \rangle = 1 \). When \( c = -1 \), \( f \) is an \( R^6 \)-valued map with \( \langle f, f \rangle = -1 \).
We choose the frame \( \{ e_A \} \) so that \( \{ e_i \} \) are tangent to \( f(M) \). In the following, the argument will be restricted to \( f(M) \). Then \( \omega^\alpha = 0 \) along \( f(M) \), and by (2.1), we have

\[
0 = - \sum_i \omega^\alpha_i \wedge \omega^i.
\]

So there exists a symmetric tensor \( \{ h^\alpha_{ij} \} \) so that

\[
\omega^\alpha_i = \sum_j h^\alpha_{ij} \omega^j,
\]

where \( h^\alpha_{ij} \) are the components of the second fundamental form \( h \) of \( f \).

In the ambient \( R^n(\supset S^5) \), \( R^5 \) or \( R^6(\supset H^5) \), according to when \( c = 1, 0 \) or \(-1\), we have

\[
d e_j = \sum_i e_i \omega^j_i + \sum_\alpha e_\alpha \omega^\alpha_j - cf \omega^j,
\]

and

\[
d e_\beta = \sum_i e_i \omega^\beta_i + \sum_\alpha e_\alpha \omega^\alpha_\beta.
\]

The mean curvature vector \( H \) of \( f \) is given by

\[
H = \frac{1}{2} \sum_\alpha (h^\alpha_{11} + h^\alpha_{22}) e_\alpha.
\]

We say that \( f \) is minimal if \( H = 0 \) identically.

3. Proof of Theorem 1.1

**Proof.** Since the ellipse of curvature is a non-degenerate circle at any point, we can choose the local orthonormal frame field \( \{ e_A \} \) so that

\[
(h^3_{ij}) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad (h^4_{ij}) = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \quad (h^5_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

where \( a > 0 \). Then

\[
\omega^3_1 = a \omega^1, \quad \omega^3_2 = -a \omega^2, \quad \omega^4_1 = a \omega^2, \quad \omega^4_2 = a \omega^1, \quad \omega^5_1 = \omega^5_2 = 0.
\]

We compute that

\[
0 = d \omega^5_1 = -\omega^3_2 \wedge \omega^4_1 - \omega^4_2 \wedge \omega^3_1 = a (\omega^1 \wedge \omega^3_2 - \omega^3_1 \wedge \omega^2)
\]

and

\[
0 = d \omega^5_2 = -\omega^3_1 \wedge \omega^4_2 - \omega^4_1 \wedge \omega^3_2 = a (\omega^2 \wedge \omega^1).
\]

Then, using the notation like

\[
\omega^5_1 = (\omega^5_1)_1 \omega^1 + (\omega^5_1)_2 \omega^2, \quad \omega^5_2 = (\omega^5_2)_1 \omega^1 + (\omega^5_2)_2 \omega^2,
\]

we have

\[
(\omega^5_1)_2 - (\omega^5_2)_1 = 0, \quad (\omega^5_1)_1 + (\omega^5_2)_2 = 0.
\]

So we can write

\[
\omega^5_1 = p \omega^1 + q \omega^2, \quad \omega^5_2 = q \omega^1 - p \omega^2
\]

for some functions \( p \) and \( q \).
For the Gauss-like map \( G = e_5 \), we have
\[ dG(e_1) = de_5(e_1) = (\omega_5^1)_1 e_3 + (\omega_5^3)_1 e_4 = -pe_3 - qe_4, \]
\[ dG(e_2) = de_5(e_2) = (\omega_5^3)_2 e_3 + (\omega_5^3)_2 e_4 = -qe_3 + pe_4, \]
and
\[ \langle dG(e_1), dG(e_1) \rangle = \langle dG(e_2), dG(e_2) \rangle = p^2 + q^2, \quad \langle dG(e_1), dG(e_2) \rangle = 0. \]
Assume that \( G \) is non-degenerate in the following. Then \( p^2 + q^2 > 0 \), and \( G \) is conformal to \( f \).

Now we have
\[ dG = -e_3(p\omega^1 + q\omega^2) - e_4(q\omega^1 - p\omega^2). \]
Let \( * \) denote the Hodge star operator so that \( *\omega^1 = \omega^2 \) and \( *\omega^2 = -\omega^1 \). Then
\[ *dG = e_3(q\omega^1 - p\omega^2) - e_4(p\omega^1 + q\omega^2) = e_3\omega_1^0 - e_4\omega_2^0. \]
We can compute that
\[ d(*dG) = -2(p^2 + q^2)e_5\omega^1 \wedge \omega^2. \]
Denoting the Laplacian by \( \Delta \), we get \( \Delta G = -2(p^2 + q^2)G \). So the Gauss-like map \( G \) is a conformal harmonic map to either \( S^5 \), \( S^4 \) or \( S^4_1 \), according to when \( c = 1, 0 \) or \(-1 \). Thus \( G \) gives a minimal surface in either \( S^5 \), \( S^4 \) or \( S^4_1 \). \( \square \)

4. Proof of Theorem 1.2

**Proof.** Since the ellipse of curvature is non-degenerate and non-circular at any point, we can choose the local orthonormal frame field \( \{ e_A \} \) so that
\[ (h_3^1) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad (h_3^2) = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}, \quad (h_3^5) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \]
where \( 0 < a < b \). We note that \( a \) and \( b \) are the semi-minor and semi-major axes of the ellipse of curvature, respectively. Then we have
\[ \omega_1^3 = a\omega^1, \quad \omega_2^3 = -a\omega^2, \quad \omega_4^3 = b\omega^2, \quad \omega_2^4 = b\omega^1, \quad \omega_5^5 = \omega_2^5 = 0. \]

We compute that
\[ d\omega_1^3 = da \wedge \omega^1 - a\omega_1^2 \wedge \omega^2 = -\omega_3^2 \wedge \omega_1^3 - \omega_3^2 \wedge \omega_1^3 = a\omega_3^2 \wedge \omega^2 - b\omega_4^2 \wedge \omega^2. \]
Using the notation like
\[ \omega_2^1 = (\omega_2^1)_1\omega^1 + (\omega_2^1)_2\omega^2, \quad \omega_3^1 = (\omega_3^1)_1\omega^1 + (\omega_3^1)_2\omega^2, \]
\[ da = a_1\omega^1 + a_2\omega^2, \quad db = b_1\omega^1 + b_2\omega^2, \]
we have
\[ 2a(\omega_2^1)_1 - b(\omega_3^1)_1 = -a_2. \]
Similarly, from \( d\omega_2^3 \), \( d\omega_4^3 \) and \( d\omega_5^4 \),
\[ 2a(\omega_2^3)_2 - b(\omega_3^4)_2 = a_1, \quad 2b(\omega_2^3)_2 - a(\omega_3^4)_2 = b_1, \quad 2b(\omega_2^3)_1 - a(\omega_3^4)_1 = -b_2. \]
Thus we get
\[ 2a\omega_2^3 - b\omega_3^4 = *da, \quad 2b\omega_2^3 - a\omega_3^4 = *db, \]
and
\[ \omega^1_2 = \frac{1}{4} (s \log(b^2 - a^2)), \quad \omega^3_4 = \frac{a(sdb) - b(sda)}{b^2 - a^2} = -s(d(a/b))/1 - (a/b)^2. \]

Next we compute that
\[ 0 = d\omega^5_1 = -\omega^5_3 \wedge \omega^5_1 - \omega^5_4 \wedge \omega^5_1 = a\omega^1 \wedge \omega^3_4 - b\omega^1_4 \wedge \omega^2 \]

and
\[ 0 = d\omega^5_2 = -\omega^5_3 \wedge \omega^5_2 - \omega^5_4 \wedge \omega^5_2 = a\omega^5_3 \wedge \omega^2 + b\omega^1 \wedge \omega^5_4. \]
Then we can write
\[ \omega^5_3 = b(p\omega^1 + q\omega^2), \quad \omega^5_5 = a(q\omega^1 - p\omega^2) \]
for some functions \( p \) and \( q \).

From \( d\omega^3_4 = -\omega^3_4 \wedge \omega^3_1 - \omega^3_2 \wedge \omega^3_1 - \omega^3_4 \wedge \omega^5_2 \), we obtain
\begin{equation}
(4.1) \quad \frac{-\Delta(a/b)}{1 - (a/b)^2} + \frac{2(a/b)|d(a/b)|^2}{(1 - (a/b)^2)^2} = ab(2 - p^2 - q^2).
\end{equation}

Set \( r = a/b \). Then \( f^r = \varepsilon \sqrt{1 - r^2} e_4 + r e_3 \). We can compute that
\[ df^r(e_1) = -\varepsilon b \sqrt{1 - r^2} e_2 + \left( \varepsilon \frac{r_2}{\sqrt{1 - r^2}} - ap \right) e_3 \]
\[ - \left( \varepsilon \frac{r_1}{\sqrt{1 - r^2}} + aq \right)(re_4 - \varepsilon \sqrt{1 - r^2} e_3), \]
and
\[ df^r(e_2) = -\varepsilon b \sqrt{1 - r^2} e_1 - \left( \varepsilon \frac{r_1}{\sqrt{1 - r^2}} + aq \right)e_3 \]
\[ - \left( \varepsilon \frac{r_2}{\sqrt{1 - r^2}} - ap \right)(re_4 - \varepsilon \sqrt{1 - r^2} e_3). \]

Set
\[ A = \varepsilon \frac{r_1}{\sqrt{1 - r^2}} + aq, \quad B = \varepsilon \frac{r_2}{\sqrt{1 - r^2}} - ap. \]
Then we have
\[ \langle df^r(e_1), df^r(e_1) \rangle = \langle df^r(e_2), df^r(e_2) \rangle = b^2 - a^2 + A^2 + B^2(> 0) \]
\[ = b^2 - a^2 + |dr|^2 - \frac{2\varepsilon a(qr_1 - pr_2)}{\sqrt{1 - r^2}} + a^2(p^2 + q^2), \]
and \( \langle df^r(e_1), df^r(e_2) \rangle = 0 \). So \( f^r \) is conformal to \( f \).

Now we have
\[ df^r = -\varepsilon b \sqrt{1 - r^2}(e_2\omega^1 + e_1\omega^2) - \varepsilon e_3(s \sin^{-1} r) - ae_3(p\omega^1 + q\omega^2) \]
\[ + \varepsilon e_4(\sqrt{1 - r^2} + e_5 dr - a e_4(q\omega^1 - p\omega^2) + e_5 \sqrt{1 - r^2} e_5(q\omega^1 - p\omega^2), \]
and
\[ + df^r = \varepsilon \sqrt{1 - r^2}(e_1\omega^2 - e_2\omega^1) + \varepsilon e_3 s (\sin^{-1} r) + e_3\omega^5_4 \]
\[ + \varepsilon e_4(s \sqrt{1 - r^2}) + e_5 s dr - r^2 e_4\omega^5_4 + e_5 \sqrt{1 - r^2} e_5\omega^5_4. \]
We need to compute $d(*df^\varepsilon)$ to get $\Delta f^\varepsilon$. We note that
\[
\Delta (\sqrt{1-r^2}) = -\frac{r \Delta r}{\sqrt{1-r^2}} - \frac{|dr|^2}{(1-r^2)^{3/2}},
\]
and by (4.1),
\[
\Delta r = ab(p^2 + q^2 - 2)(1-r^2) - \frac{2r|dr|^2}{1-r^2}.
\]
By a little long but straight computation, we can show that
\[
\Delta f^\varepsilon = -2\left(b^2 - a^2 + \frac{|dr|^2}{1-r^2} + \frac{2sa(qr_1 - pr_2)}{\sqrt{1-r^2}} + a^2(p^2 + q^2)\right)f^\varepsilon.
\]
Hence, the map $f^\varepsilon$ is a conformal harmonic map to either $S^5$, $S^4$ or $S^5_1$, according to when $c = 1$, 0 or $-1$. Thus $f^\varepsilon$ gives a minimal surface in either $S^5$, $S^4$ or $S^5_1$. □

References