APPLICATIONS OF \((p, q)\)-GAMMA FUNCTION 
TO SZÁSZ DURRMEYER OPERATORS

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Abstract. We define a \((p, q)\) analogue of Gamma function. As an application, 
we propose \((p, q)\)-Szász–Durrmeyer operators, estimate moments and establish 
some direct results.

1. Introduction

In the last two decades the \textit{quantum calculus} is an active area of research 
among researchers. The \textit{quantum calculus} find applications in a number of areas, 
including approximation theory. The relationship between approximation theory 
and \textit{q-calculus} encouraged the mathematicians to give \textit{q-analogue} of known results 
(see \cite{3}). This rapid development of \textit{q-calculus} has led to the discovery of new 
generalization of this theory. This produces some advantages like that the rate 
of convergence of \(q\)-operators is more flexible and better than the classical one. 
Since the \textit{q-calculus} is based on one parameter, there is a possibility of extension 
of \textit{q-calculus}. In this direction Sahai–Yadav \cite{14} established some extensions to 
post-quantum calculus in special functions. A question arises: can we modify the 
operators using \((p, q)\)-calculus such that our modified operator has better error 
estimation than the classical ones. For this purpose, we will define \((p, q)\)-Szász–
Durrmeyer operators. Several well-known operators may extend to \((p, q)\)-\textit{analogue}s. 
Mursaleen et al introduced the \((p, q)\)-\textit{analogue} of the Bernstein operators in \cite{10}. 
There both point-wise convergence and asymptotic formula are considered. Other 
important class of discrete operators has been investigated by using \((p, q)\)-\textit{calculus}. 
For example \((p, q)\)-Bernstein–Stancu operators appeared in \cite{9} \((p, q)\) Bleimann–
Butzer–Hahn and \((p, q)\)-Szász Mirakyan operators have been studied recently in \cite{1,11}. Very recently, in order to obtain an approximation process in the space of 
\((p, q)\)-Bernstein operators, the authors \cite{5} defined Durrmeyer type modification of 
\((p, q)\)-Bernstein operators.

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Motivated by all the above results we propose Durrmeyer type modification of the \((p, q)\)-Szász Mirakyan operators using an integral version of \((p, q)\)-Gamma function (as we know it is first in literature).

The paper is organized as follows: the next section contains some basic facts regarding \((p, q)\)-calculus, we also introduce \((p, q)\)-analogue of Gamma function. The construction of the announced class of operators is presented in Section 3. Section 4 deals with the quantitative type estimate with a suitable modulus of continuity. The last section is devoted to weighted Korovin type theorems and we estimate the approximation of bounded functions by announced operators with the help of a Lipschitz-type maximal function.

2. Notations and Preliminaries

Following the definitions and notations of [14]:
Set \(N_0 = \{0\} \cup \mathbb{N}\), the \((p, q)\)-numbers are defined as
\[
[n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} = \frac{p^n - q^n}{p - q}
\]
for \(n \in \mathbb{N}\). The \((p, q)\)-factorial \([n]_{p,q}!\) of the element \(n \in \mathbb{N}\) means
\[
[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}, n \geq 1, [0]_{p,q}! = 1.
\]

The \((p, q)\)-binomial theorem is given by
\[
\begin{align*}
1 \Phi_0((a, b); -; (p, q)) & = \frac{((p, b) x; -; (p, q))_\infty}{((p, a) x; -; (p, q))_\infty}, \\
E_{p,q}(x) & = \sum_{n=0}^{\infty} \frac{p^n(n-1)/2}{[n]_{p,q}!} x^n = 1 \Phi_0((1, 0); -; (p, q), x), \\
e_{p,q}(x) & = \sum_{n=0}^{\infty} \frac{q^n(n-1)/2}{[n]_{p,q}!} x^n = 1 \Phi_0((0, 1); -; (p, q), -x).
\end{align*}
\]

We know that \(1 \Phi_0((1, 0); -; (p, q), x) 1 \Phi_0((0, 1); -; (p, q), x) = 1\), that is the following relation between \((p, q)\)-exponential functions
\[
(2.2) \quad e_{p,q}(x)E_{p,q}(-x) = 1
\]
holds. We mention that these \((p, q)\)-analogues of the classical exponential functions are valid for \(0 < q < p \leq 1\). Moreover \(E_{p,q}(x)\) and \(e_{p,q}(x)\) tend to \(e^x\) as \(p \to 1^-\) and \(q \to 1^-\).

It is obvious by the \((p, q)\)-derivative formula \(D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, x \neq 0\) that
\[
\begin{align*}
D_{p,q}E_{p,q}(x) & = E_{p,q}(qx), \\
D_{p,q}E_{p,q}(ax) & = aE_{p,q}(aqx).
\end{align*}
\]

Proposition 2.1. The formula of \((p, q)\)-integration by part is given by
\[
\int_a^b f(px)D_{p,q}g(x) \, dp_q x = f(b)g(b) - f(a)g(a) - \int_0^a g(qx)D_{p,q}f(x) \, dp_q x
\]

Definition 2.1. For any \(n \in \mathbb{N}\), we define a \((p, q)\)-Gamma function by
\[
\Gamma_{p,q}(n) = \int_0^\infty p^{(n-1)(n-2)/2}x^{n-1}E_{p,q}(-qx) \, dp_q x.
\]

Lemma 2.1. For any \(n \in \mathbb{N}\), we have \(\Gamma_{p,q}(n+1) = [n]_{p,q}!\).

Proof. From (2.1) we have \(E_{p,q}(0) = 1\) and from (2.2) we have
\[
E_{p,q}(\infty) = \lim_{x \to \infty} E_{p,q}(x) = \lim_{x \to \infty} e_{p,q}(-x) = \lim_{x \to \infty} \Phi_0((1,0); -; (p,q), -x)
\]
\[
= \lim_{x \to \infty} ((p,0); -; (p,q)) = 0.
\]
Also from (2.3) we can write
\[
\Gamma_{p,q}(n+1) = \int_0^\infty p^{(n-1)/2}x^nE_{p,q}(-qx) \, dp_q x.
\]

By Proposition 2.1 using \((p, q)\)-integration by parts for \(f(x) = x^n\) and \(g(x) = E_{p,q}(-x)\), we have
\[
\Gamma_{p,q}(n+1) = [n]_{p,q} \int_0^\infty p^{(n-1)/2}x^{n-1}E_{p,q}(-qx) \, dp_q x
\]
\[
= [n]_{p,q} \int_0^\infty p^{(n-1)/2}x^{n-1}D_{p,q}E_{p,q}(-x) \, dp_q x.
\]

Thus, we have
\[
\Gamma_{p,q}(n+1) = [n]_{p,q} \Gamma_{p,q}(n) = [n]_{p,q}[n - 1]_{p,q} \Gamma_{p,q}(n - 1) = [n]_{p,q}!.
\]

An alternate form of \((p, q)\)-Gamma function without integral expression for \(n\) nonnegative integer, is given in 12 by
\[
\Gamma_{p,q}(n+1) = \frac{(p \odot q)^n}{(p - q)^n} = [n]_{p,q}!, \quad 0 < q < p.
\]

3. \((p, q)\)-Szász–Durrmeyer Operators and Moments

In order to introduce a \((p, q)\) Durrmeyer variant for Szasz–Mirakyan operators, we present a construction due to Acar 1. The \((p, q)\)-analogue of Szasz operators for \(x \in [0, \infty)\) and \(0 < q < p \leq 1\) defined by in the following way
\[
S_{n,p,q}(f; x) = \sum_{k=0}^{n} s_{n,k}^p(x) f\left(\frac{[k]_{p,q}}{q^{k-2}[n]_{p,q}}\right),
\]
where

\[ s_{n,k}^{p,q}(x) = \frac{1}{E_{p,q}([n]_{p,q}x)} \frac{q^{k(k-1)/2}}{[k]_{p,q}} \langle [n]_{p,q}x \rangle^k. \]

In case \( p = 1 \), we get the \( q \)-Szász operators [2]. If \( p = q = 1 \), we get at once the well known Szász operators.

**Lemma 3.1.** [1] For \( x \in [0, \infty) \), \( 0 < q < p \leq 1 \), we have

1. \( S_{n,p,q}(1;x) = 1 \),
2. \( S_{n,p,q}(t; x) = qx \),
3. \( S_{n,p,q}(t^2; x) = pqx^2 + \frac{q^2}{p}x \).

The Szász operators defined by (3.1) are discrete operators. The integral modification of these operators was proposed in [7]. Different variants and \( q \)-analogues in [3] and [6]. As an application of the \((p,q)\)-Gamma function, we introduce below the Durrmeyer type \((p,q)\) variant of the Szász operators as

**Definition 3.1.** The \((p,q)\)-analogue of Szász–Durrmeyer operator for \( x \in [0, \infty) \) and \( 0 < q < p \leq 1 \) is defined by

\[ \tilde{S}_{n,p,q}(f; x) = \left[ [n]_{p,q} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \right] \int_0^\infty p^{k(k-1)/2} \frac{([n]_{p,q}t)^k}{[k]_{p,q}!} E_{p,q}(-q[n]_{p,q}t)f(q^{1-k}pt) \, dp \, dt \]

where \( s_{n,k}^{p,q}(x) \) is defined in (3.1).

It may be remarked here that for \( p = q = 1 \) these operators reduces to the Szász–Durrmeyer operators.

**Lemma 3.2.** For \( x \in [0, \infty) \), \( 0 < q < p \leq 1 \), we have

1. \( \tilde{S}_{n,p,q}(1; x) = 1 \),
2. \( \tilde{S}_{n,p,q}(t; x) = \frac{x}{[n]_{p,q}} + px \)
3. \( \tilde{S}_{n,p,q}(t^2; x) = \frac{p^2}{q}x^2 + \frac{|x|^2}{[n]_{p,q}} + \frac{|2|_{p,q}^2}{[p]^2[n]_{p,q}} \).

**Proof.** Using Definition 2.1, Lemmas 2.1 and 3.1 we have

\[ \tilde{S}_{n,p,q}(1; x) = \left[ [n]_{p,q} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \right] \int_0^\infty p^{k(k-1)/2} \frac{([n]_{p,q}t)^k}{[k]_{p,q}!} E_{p,q}(-q[n]_{p,q}t) \, dp \, dt \]

\[ = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{\Gamma_p(q + 1)}{[k]_{p,q}!} = 1 \]

and next using \([k + 1]_{p,q} = q^k + p[k]_{p,q}\), we have

\[ \tilde{S}_{n,p,q}(t; x) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \int_0^\infty p^{k(k-1)/2} q^{1-k} \frac{([n]_{p,q}t)^{k+1}}{[k]_{p,q}!} E_{p,q}(-q[n]_{p,q}t) \, dp \, dt \]

\[ = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) q^{1-k} \frac{\Gamma_p(q + 2)}{[k]_{p,q}! [n]_{p,q}} = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) q^{1-k} \frac{[k + 1]_{p,q}}{[n]_{p,q}} \]

\[ = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) q^{1-k} \frac{(q^k + p[k]_{p,q})}{[n]_{p,q}} \]
For \( f \) functions on 
and
\[
\tilde{S}_{n,p,q}(t^2; x) = \frac{1}{[n]_{p,q}^2} \sum_{k=0}^{\infty} s^p,q_{n,k}(x) \int_0^\infty \frac{p^{k+2} t^{k+2}}{[k]_{p,q}^2} E_{p,q}(-q[x]_{p,q}t) \, dt
\]

and
\[
\tilde{S}_{n,p,q}(t^2; x) = \frac{1}{[n]_{p,q}^2} \sum_{k=0}^{\infty} s^p,q_{n,k}(x) \int_0^\infty \frac{p^{k+2} t^{k+2}}{[k]_{p,q}^2} E_{p,q}(-q[x]_{p,q}t) \, dt
\]

\[
[3.2] \quad \tilde{S}_{n,p,q}(t^2; x) = \frac{p^3 x^2}{q} + \frac{[2]_{p,q}^2 x}{[n]_{p,q}} + \frac{[2]_{p,q}^2 q^2}{[p]_{p,q}^2}
\]

Remark 3.1. For \( 0 < q < p \leq 1 \) we may write
\[
\tilde{S}_{n,p,q}(t^2; x) = \frac{q}{[n]_{p,q}} + (p - 1)x.
\]

4. Quantitative Estimate

By \( C_B[0, \infty) \) we denote the class of all real valued continuous and bounded functions on \([0, \infty)\). The norm \( ||f||_B \) is defined by
\[
||f||_B = \sup_{x \in [0, \infty)} |f(x)|.
\]

For \( f \in C_B \) the Steklov mean is defined as
\[
(4.1) \quad f_h(x) = \frac{4}{h^2} \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} [2f(x + u + v) - f(x + 2(u + v))] \, du \, dv
\]
By simple computation, it is observed that
(i) \( \|f_h - f\|_{C_B} \leq \omega_2(f, h) \).
(ii) If \( f \) is continuous, then \( f_h', f'' \in C_B \) and
\[
\|f_h'\|_{C_B} \leq \frac{5}{h} \omega(f, h), \quad \|f_h''\|_{C_B} \leq \frac{9}{h^2} L \omega_2(f, h),
\]
where the first and second order moduli of continuity are respectively defined by
\[
\omega(f, \delta) = \sup_{x, u, v \geq 0, |u-v| \leq \delta} |f(x + u) - f(x + v)|,
\]
\[
\omega_2(f, \delta) = \sup_{x, u, v \geq 0, |u-v| \leq \delta} |f(x + 2u) - 2f(x + u + v) + f(x + 2v)|, \quad \delta \geq 0.
\]

**Theorem 4.1.** Let \( q \in (0, 1) \) and \( p \in (q, 1] \) The operator \( \tilde{S}_{n,p,q} \) maps space \( C_B \) into \( C_B \) and \( \|\tilde{S}_{n,p,q}(f)\|_{C_B} \leq \|f\|_{C_B} \).

**Proof.** Let \( q \in (0, 1) \) and \( p \in (q, 1] \). From Lemma 3.2 we have
\[
|\tilde{S}_{n,p,q}(f, x)| \leq [n]_{p,q} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \int_0^\infty p^{(k-1)/2} \left( \frac{[n]_{p,q} t^k}{[k]_{p,q}} \right) E_{p,q}(-q[n]_{p,q} t) d_{p,q} t
\]
\[
\leq \sup_{x \in [0, \infty)} |f(x)| [n]_{p,q} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \int_0^\infty p^{(k-1)/2} \left( \frac{[n]_{p,q} t^k}{[k]_{p,q}} \right) E_{p,q}(-q[n]_{p,q} t) d_{p,q} t
\]
\[
= \sup_{x \in [0, \infty)} |f(x)| \|\tilde{S}_{n,p,q}(1, x)\|_{C_B}.
\]

We are going to study the degree of approximation in terms of the first and second order moduli of continuity.

**Theorem 4.2.** Let \( q \in (0, 1) \) and \( p \in (q, 1] \). If \( f \in C_B[0, \infty) \), then
\[
|\tilde{S}_{n,p,q}(f, x) - f(x)| \leq 5\omega \left( f, \frac{1}{\sqrt{[n]_{p,q}}} \right) \left( \frac{q}{\sqrt{[n]_{p,q}}} + \sqrt{[n]_{p,q}(p-1)x} \right)
\]
\[
+ \frac{9}{2} \omega_2 \left( f, \frac{1}{\sqrt{[n]_{p,q}}} \right) \left[ 2 + \frac{(p^3 - 2pq + q)[n]_{p,q} x^2}{q} + \frac{[2]_{p,q}^2 q^2}{p[n]_{p,q}} x^2 \right].
\]

**Proof.** For \( x \geq 0 \) and \( n \in \mathbb{N} \) and using the Steklov mean \( f_h \) defined by (4.1), we can write
\[
|\tilde{S}_{n,p,q}(f, x) - f(x)| \leq \tilde{S}_{n,p,q}(|f - f_h|, x) + |\tilde{S}_{n,p,q}(f_h - f_h(x), x)| + |f_h(x) - f(x)|.
\]
First by Theorem 4.1 and property (i) of the Steklov mean we have
\[
\tilde{S}_{n,p,q}(|f - f_h|, x) \leq \|\tilde{S}_{n,p,q}(f - f_h)|_{C_B} \leq \|f - f_h\|_{C_B} \leq \omega_2(f, h).
\]
By Lemma 3.2, we have

\[ |\tilde{S}_{n,p,q}(f_n - f_n(x), x)| \leq |f'_n(x)|\tilde{S}_{n,p,q}(t - x, x) + \frac{1}{2}||f''||_{C_n}\tilde{S}_{n,p,q}((t - x)^2, x). \]

So, for sufficiently large \( n \), we have

\[ |\tilde{S}_{n,p,q}(f_n - f_n(x), x)| \leq \frac{5}{6} \omega(f, h) \left( \frac{q}{p} + (p - 1)x \right) + \frac{9}{2h^2}\omega_2(f, h)\tilde{S}_{n,p,q}((t - x)^2, x), \]

where \( \tilde{S}_{n,p,q}((t - x)^2, x) \) is given by (3.2). For \( x \geq 0, \ h > 0 \) and choosing \( h = \sqrt{1/[n]_{p,q}} \), we get the desired result. \( \square \)

**Remark 4.1.** For \( q \in (0, 1) \) and \( p \in (q, 1) \) it is seen that \( \lim_{n \to \infty}[n]_{p,q} = 1/(q - p) \). In order to consider the convergence of \((p, q)\)-Szász–Durrmeyer operators, we assume \( p = (p_n) \) and \( q = (q_n) \) such that \( 0 < q_n < p_n \leq 1 \) and for \( n \) sufficiently large \( p_n \to 1, q_n \to 1, p'_n \to a, q'_n \to b \), so that \([n]_{p_n,q_n} \to \infty\). Such a sequence can always be constructed for example, we can take \( p_n = 1 - 1/2n \) and \( q_n = 1 - 1/n \). Clearly \( \lim_{n \to \infty} p'_n = e^{-1/2}, \lim_{n \to \infty} q'_n = e^{-1} \) and \( \lim_{n \to \infty}[n]_{p_n,q_n} = \infty \).

5. Direct Estimates

Let us denote by \( H_{+}[0, \infty) \) the set of all functions \( f \) defined on the positive real axis satisfying the condition \( |f(x)| \leq M_f(1 + x^2) \), where \( M_f \) is an absolute constant depending on \( f \). By \( C_z[0, \infty) \), we mean the subspace of all continuous functions belonging to \( H_{+}[0, \infty) \). Also, let \( C_z^*[0, \infty) \) denote the subspace of all functions \( f \in C_z[0, \infty) \), for which \( \lim_{|x| \to \infty} f(x)/(1 + x^2) \) is finite. The class \( C_z^*[0, \infty) \) is endowed with the norm

\[ ||f||_{z^2} = \sup_{x \in [0, \infty)} \frac{|f(x)|}{1 + x^2}. \]

We discuss below the weighted approximation theorem, where the approximation formula is valid for the positive real axis (see [4]).

**Theorem 5.1.** Let \( p = p_n \) and \( q = q_n \) satisfies \( 0 < q_n < p_n \leq 1 \) and for \( n \) sufficiently large \( p_n \to 1, q_n \to 1 \) and \( p'_n \to a, q'_n \to b \). For each \( f \in C_z^*[0, \infty) \), we have \( \lim_{n \to \infty} ||\tilde{S}_{n,p_n,q_n}(f) - f||_{z^2} = 0 \).

**Proof.** Using Korovkin’s theorem, it is sufficient to verify the following three conditions

\[ \lim_{n \to \infty} ||\tilde{S}_{n,p_n,q_n}(t^{\nu}, x) - x^{\nu}||_{z^2} = 0, \ \nu = 0, 1, 2. \]

Since \( \tilde{S}_{n,p_n,q_n}(1, x) = 1 \) the first condition of (5.1) is fulfilled for \( \nu = 0 \).

For \( n \in \mathbb{N} \), we can write,

\[ ||\tilde{S}_{n,p_n,q_n}(t, x) - x||_{z^2} \leq \frac{q_n}{[n]_{p_n,q_n}} + (p_n - 1) \sup_{x \in [0, \infty)} \frac{x}{1 + x^2}. \]
\[ \|S_{n,p,q_n}(t^2, x) - x^2\|_{2} \leq \left( \frac{p_n^3}{q_n} - 1 \right) \sup_{x \in [0, \infty)} \frac{x^2}{1 + x^2} + \sup_{x \in [0, \infty)} \frac{x}{1 + x^2} \]

which implies that for \( v = 1, 2 \) we have \( \lim_{n \to \infty} \|S_{n,p,q_n}(t^v, x) - x^v\|_{2} = 0 \). \( \square \)

We give the following theorem to approximate all functions in \( C_{\infty}[0, \infty) \).

**Theorem 5.2.** Let \( p = p_n \) and \( q = q_n \) satisfies \( 0 < q_n < p_n \leq 1 \) and for \( n \) sufficiently large \( p_n \to 1, q_n \to 1 \) and \( q_n^a \to a \) and \( p_n^a \to b \). For each \( f \in C_{\infty}[0, \infty) \) and \( \alpha > 0 \), we have

\[ \lim_{n \to \infty} \sup_{x \in [0, \infty)} \frac{|S_{n,p,q_n}(f, x) - f(x)|}{(1 + x^2)^{1+\alpha}} = 0. \]

**Proof.** For any fixed \( x_0 > 0 \),

\[ \begin{align*}
&\sup_{x \in [0, \infty)} \frac{|S_{n,p,q_n}(f, x) - f(x)|}{(1 + x^2)^{1+\alpha}} \\
&= \sup_{x < x_0} \frac{|S_{n,p,q_n}(f, x) - f(x)|}{(1 + x^2)^{1+\alpha}} + \sup_{x > x_0} \frac{|S_{n,p,q_n}(f, x) - f(x)|}{(1 + x^2)^{1+\alpha}} \\
&\leq \|S_{n,p,q_n}(f) - f\|_{C[0, a]} + \|f\|_{2} \sup_{x > x_0} \frac{|S_{n,p,q_n}(1 + t^2, x)|}{(1 + x^2)^{1+\alpha}} + \sup_{x > x_0} \frac{|f(x)|}{(1 + x^2)^{1+\alpha}}.
\end{align*} \]

By Lemma 3.2 and the well known Korovkin theorem, the first term of the above inequality tends to zero for a sufficiently large \( n \). By Lemma 3.2 for any fixed \( x_0 > 0 \), it is easily seen that \( \sup_{x > x_0} \frac{|S_{n,p,q_n}(1 + t^2, x)|}{(1 + x^2)^{1+\alpha}} \) tends to zero as \( n \to \infty \). We can choose \( x_0 > 0 \) so large that the last part of above inequality can be made small enough. This completes the proof of the theorem. \( \square \)

Now we establish some point-wise estimates of the rate of convergence of \((p,q)\)-Szász–Durrmeyer operators. First, we give the relationship between the local smoothness of \( f \) and local approximation. A function \( f \in C(0, \infty) \) is said to satisfy the Lipschitz condition \( \text{Lip}_\alpha \) on \( D, \alpha \in (0, 1], D \subset [0, \infty) \) if

\[ |f(t) - f(x)| \leq M_f |t - x|^{\alpha}, \quad t \in [0, \infty) \text{ and } x \in D, \]

where \( M_f \) is a constant depending only \( \alpha \) and \( f \).

**Theorem 5.3.** Let \( f \in \text{Lip}_\alpha \) on \( D, D \subset [0, \infty) \) and \( \alpha \in (0, 1] \). We have

\[ |S_{n,p,q}(f, x) - f(x)| \leq \left( \frac{p^3 - 2pq + q}{q} \right) x^2 + \frac{(2|p, q|^2 - 2q)x}{|p, q|} + \frac{|p, q|^2}{p|p, q|^2} x^{\alpha/2} + 2d^\alpha(x; D) \]

where \( d(x; D) \) represents the distance between \( x \) and \( D \).
We have
\[|f(t) - f(x)| \leq |f(t) - f(x_0)| + |f(x_0) - f(x)|, \quad x \in [0, \infty).\]
Using (3.2), we get
\[|\tilde{S}_{n,p,q}(f, x) - f(x)| \leq \tilde{S}_{n,p,q}(|f(t) - f(x_0)|, x) + |f(x_0) - f(x)| \leq M_f \tilde{S}_{n,p,q}(|t - x_0|^\alpha, x) + M_f |x_0 - x|^\alpha.\]
Then, with Hölder’s inequality with \( p := \frac{2}{\alpha} \) and \( \frac{1}{r} := 1 - \frac{1}{p}, \) we have
\[\tilde{S}_{n,p,q}(|t - x|^\alpha, x) \leq (\tilde{S}_{n,p,q}(|t - x|^2, x))^{\frac{\alpha}{2}} (\tilde{S}_{n,p,q}(1, x))^{1 - \frac{\alpha}{2}}.\]
Also, \( \tilde{S}_{n,p,q} \) is monotone
\[\tilde{S}_{n,p,q}(|t - x|^\alpha, x) \leq (\tilde{S}_{n,p,q}(|t - x|^\alpha, x))^{\frac{\alpha}{2}} + |x_0 - x|^\alpha.\]
Using (5.3), (5.4) and (3.2), we get the desired result.

Proof. For \( x_0 \in \bar{D}, \) the closure of the set \( D \) in \([0, \infty),\) we have
\[|f(t) - f(x)| \leq |f(t) - f(x_0)| + |f(x_0) - f(x)|, \quad x \in [0, \infty).\]
Applying Hölder’s inequality with \( p := \frac{2}{\alpha} \) and \( \frac{1}{r} := 1 - \frac{1}{p}, \) we have
\[\tilde{S}_{n,p,q}(|t - x|^\alpha, x) \leq (\tilde{S}_{n,p,q}(|t - x|^2, x))^{\frac{\alpha}{2}} (\tilde{S}_{n,p,q}(1, x))^{1 - \frac{\alpha}{2}}.\]
Using (3.2), we have our assertion.

**References**

2. A. Aral, V. Gupta, The \( q \)-derivative and applications to \( q \)-Szász Mirakyan operators, Calcolo 43(3) (2006), 151–170.