KOROVKIN TYPE THEOREM FOR FUNCTIONS OF TWO VARIABLES VIA LACUNARY EQUISTATISTICAL CONVERGENCE

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Abstract. Aktuğlu and Gezer [1] introduced the concepts of lacunary equistatistical convergence, lacunary statistical pointwise convergence and lacunary statistical uniform convergence for sequences of functions. Recently, Kaya and Gönül [11] proved some analogs of the Korovkin approximation theorem via lacunary equistatistical convergence by using test functions $1, \frac{x}{1+x}, \frac{y}{1+y}, (\frac{x}{1+x})^2 + (\frac{y}{1+y})^2$. We apply the notion of lacunary equistatistical convergence to prove a Korovkin type approximation theorem for functions of two variables by using test functions $1, \frac{x}{1-x}, \frac{y}{1-y}, (\frac{x}{1-x})^2 + (\frac{y}{1-y})^2$.

1. Introduction and preliminaries

The following concept of statistical convergence for sequences of real numbers was introduced by Fast [6]. Let $K \subseteq \mathbb{N}$ and $K_n = \{j : j \leq n, j \in K\}$. Then the natural density of $K$ is defined by $\delta(K) := \lim_{n \to \infty} |K_n|/n$ if the limit exists, where $|K_n|$ denotes the cardinality of the set $K_n$.

A sequence $x = (x_j)$ of real numbers is said to be statistically convergent to the number $L$ if, for every $\epsilon > 0$, the set $\{j : j \in \mathbb{N}, |x_j - L| \geq \epsilon\}$ has natural density zero, that is, if, for each $\epsilon > 0$, we have

$$\lim \frac{1}{n} \left| \{j : j \leq n, |x_j - L| \geq \epsilon\} \right| = 0.$$ 

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by $\theta$ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio $k_r/k_{r-1}$ will be abbreviated by $q_r$.

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Fridy and Orhan [7] defined the notion of lacunary statistical convergence as follows. Let $\theta$ be a lacunary sequence; the number sequence $x$ is $S_\theta$-convergent to $L$ provided that for every $\epsilon > 0$,

$$\lim_{r \to \infty} \frac{1}{h_r} \left| \{ k \in I_r : |x_k - L| \geq \epsilon \} \right| = 0.$$ 

In this case we write $S_\theta$-limit $x = L$ or $x_k \to L (S_\theta)$.

The concept of equistatistical convergence was introduced by Balcerzak et al. [2] and was subsequently applied for deriving approximation theorems in [13][18][19]. In [1], Aktuglu and Gezer [1] generalized the idea of statistical convergence to lacunary equistatistical convergence. Recently, Kaya and Gönül [11] established some analogs of the Korovkin approximation theorem via lacunary equistatistical convergence. Korovkin type approximation theorems for various kinds of statistical convergence are studied in [3][5][14][18]. In this paper, we prove such type of theorem via lacunary equistatistical convergence by using the test functions $1$, $x_1 - x$ and $(x_1 - x)^2$.

Let $C[a, b]$ be the linear space of all real-valued continuous functions $f$ on $[a, b]$. We know that $C[a, b]$ is a Banach space with the norm given by

$$\|f\|_{C[a, b]} := \sup_{x \in [a, b]} |f(x)| \quad (f \in C[a, b]).$$

Let $f$ and $f_n (n \in \mathbb{N})$ be real-valued functions defined on a subset $X$ of the set $\mathbb{N}$ of positive integers.

**Definition 1.1.** A sequence $(f_k)$ of real-valued functions is said to be lacunary equi-statistically convergent to $f$ on $X$ if, for every $\epsilon > 0$, the sequence $(S_r(\epsilon, x))$ converges uniformly to the zero function on $X$, that is, if, for every $\epsilon > 0$, we have $\lim_{r \to \infty} \|S_r(\epsilon, x)\|_{C(X)} = 0$, where

$$S_r(\epsilon, x) := \frac{1}{h_r} \left| \{ k : k \in I_r, |f_k(x) - f(x)| \geq \epsilon \} \right|$$

and $C(X)$ denotes the space of all continuous functions on $X$. In this case, we write

$$f_k \Rightarrow f \quad (\theta\text{-equistat}).$$

**Definition 1.2.** A sequence $(f_k)$ is said to be lacunary statistically pointwise convergent to $f$ on $X$ if, for every $\epsilon > 0$ and for each $x \in X$, we have

$$\lim_{r \to \infty} \frac{1}{h_r} \left| \{ k : k \in I_r, |f_k(x) - f(x)| \geq \epsilon \} \right| = 0.$$ 

In this case, we write $f_r \to f \quad (\theta\text{-stat}).$

**Definition 1.3.** A sequence $(f_r)$ is said to be lacunary statistically uniformly convergent to $f$ on $X$ if (for every $\epsilon > 0$), we have

$$\lim_{r \to \infty} \frac{1}{h_r} \left| \{ k : k \in I_r, \|f_k - f\|_{C(X)} \geq \epsilon \} \right| = 0.$$ 

In this case, we write $f_r \Rightarrow f \quad (\theta\text{-stat}).$
Definition 1.4. (see [10]). A sequence \((f_n)\) of real-valued functions is said to be equistatistically convergent to \(f\) on \(X\) if, for every \(\epsilon > 0\), the sequence \((P_{n,\epsilon}(x))_{n\in\mathbb{N}}\) of real-valued functions converges uniformly to the zero function on \(X\), that is, if (for every \(\epsilon > 0\)) we have \(\lim_{n\to\infty} \|P_{n,\epsilon}(x)\|_{C(X)} = 0\), where

\[ P_{n,\epsilon}(x) = \frac{1}{n} \left\{ k : k \leq n, |f_k(x) - f(x)| \geq \epsilon \right\} = 0. \]

In this case, we write \(f_k \rightsquigarrow f\) (equistat).

The following implications of the above definitions and concepts are trivial.

\[ f_k \Rightarrow f \ (\theta\text{-stat}) \Rightarrow f_k \rightsquigarrow f \ (\theta\text{-equistat}) \Rightarrow f_k \to f \ (\theta\text{-stat}). \]

Furthermore, in general, the reverse implications do not hold true.

2. Main Results

Let \(I = [0, A], J = [0, B]\), \(A, B \in (0, 1)\) and \(K = I \times J\). We denote by \(C(K)\) the space of all continuous real valued functions on \(K\). This space is equipped with the norm \(\|f\|_{C(K)} := \sup_{(x,y) \in K} |f(x,y)|\), \(f \in C(K)\). Let \(H_\omega(K)\) denote the space of all real valued functions \(f\) on \(K\) such that

\[ |f(s,t) - f(x,y)| \leq \omega(f, \sqrt{\left(\frac{s}{1-s} - \frac{x}{1-x}\right)^2 + \left(\frac{t}{1-t} - \frac{y}{1-y}\right)^2}), \]

where \(\omega\) is the modulus of continuity, i.e.

\[ \omega(f; \delta) = \sup_{(s,t), (x,y) \in K} \{ |f(s,t) - f(x,y)| : \sqrt{(s-x)^2 + (t-y)^2} \leq \delta \}. \]

It is to be noted that any function \(f \in H_\omega(K)\) is continuous and bounded on \(K\).

In [1], Aktuğlu and Gezer proved the Korovkin theorem for lacunary equistatistical convergence by using the test functions \(1, x\) and \(x^2\); while Kaya and Gönil [11] used the test functions \(1, \frac{x}{1+x}, \left(\frac{y}{1+y}\right)^2, \left(\frac{x}{1+y}\right)^2 + \left(\frac{y}{1+y}\right)^2\). Recently, Srivastava et al. [19] defined and studied the \(\lambda\)-equistatistical convergence of positive linear operators by using the notion of \(\lambda\)-statistical convergence [15]. In this paper, we apply the notion of lacunary equistatistical convergence to prove a Korovkin type approximation theorem for functions of two variables by using test functions \(1, \frac{x}{1-x}, \left(\frac{y}{1+y}\right)^2, \left(\frac{x}{1+y}\right)^2 + \left(\frac{y}{1+y}\right)^2\).

Let \(T\) be a linear operator which maps \(C[a,b]\) into itself. We say that \(T\) is positive if, for every non-negative \(f \in C[a,b]\), we have \(T(f,x) \geq 0\) \((x \in [a,b])\).

We prove the following result:

**THEOREM 2.1.** Let \(\theta = (k_r)\) be a lacunary sequence, and let \((L_r)\) be a sequence of positive linear operators from \(H_\omega(K)\) into \(C_B(K)\). Then for all \(f \in H_\omega(K)\)

\[ L_r(f; x, y) \rightsquigarrow f(x, y) \ (\theta\text{-equistat}) \]

if and only if

\[ L_r(f; x, y) \rightsquigarrow g_i(x, y) \ (\theta\text{-equistat}) \ (i = 0, 1, 2, 3), \]

with \(g_0(x) = 1\), \(g_1(x) = \frac{x}{1-x}\), \(g_2(x) = \frac{x}{1+y}\) and \(g_3(x) = \left(\frac{x}{1-y}\right)^2 + \left(\frac{y}{1+y}\right)^2\).
Proof. Since each of the functions \( f_i \) belongs to \( H_\omega(K) \), conditions (2.2) follow immediately. Let \( g \in H_\omega(K) \) and \((x, y) \in K\) be fixed. Then for \( \varepsilon > 0 \), there exist \( \delta_1, \delta_2 > 0 \) such that \( |f(s, t) - f(x, y)| < \varepsilon \) holds for all \((s, t) \in K\) satisfying
\[
\left| \frac{s}{1-s} - \frac{x}{1-x} \right| < \delta_1, \quad \left| \frac{t}{1-t} - \frac{y}{1-y} \right| < \delta_2.
\]
Let
\[
K(\delta_1, \delta_2) := \left\{(s, t) \in K : \left| \frac{s}{1-s} - \frac{x}{1-x} \right| < \delta_1, \left| \frac{t}{1-t} - \frac{y}{1-y} \right| < \delta_2 \right\}.
\]
Hence
\[
(2.3) \quad |f(s, t) - f(x, y)| = |f(s, t) - f(x, y)|_{\chi_K(\delta_1, \delta_2)}(s, t) + |f(s, t) - f(x, y)|_{\chi_K \cap K(\delta_1, \delta_2)}(s, t) \leq \varepsilon + 2N_{\chi_K \cap K(\delta_1, \delta_2)}(s, t),
\]
where \( \chi_D \) denotes the characteristic function of the set \( D \) and \( N = \|f\|_{C_\alpha(K)} \).

Further we get
\[
\chi_{K \cap K(\delta_1, \delta_2)}(s, t) \leq \frac{1}{\delta_1^2} \left( \frac{s}{1-s} - \frac{x}{1-x} \right)^2 + \frac{1}{\delta_2^2} \left( \frac{t}{1-t} - \frac{y}{1-y} \right)^2.
\]
Combining (2.3) and (2.4) and choosing \( \delta := \min\{\delta_1, \delta_2\} \), we get
\[
|f(s, t) - f(x, y)| \leq \varepsilon + \frac{2N}{\delta^2} \left( \left( \frac{s}{1-s} - \frac{x}{1-x} \right)^2 + \left( \frac{t}{1-t} - \frac{y}{1-y} \right)^2 \right).
\]
After using the linearity and positivity of operators \{\( L_r \)\}, we get
\[
|L_r(f; x, y) - f(x, y)| \leq \varepsilon + M \left\{|L_r(g_0; x, y) - g_0(x, y)| + |L_r(g_1; x, y) - g_1(x, y)| \right. \\
\left. + |L_r(g_2; x, y) - g_2(x, y)| + |L_r(g_3; x, y) - g_3(x, y)|\right\},
\]
which implies that
\[
|L_r(f; x, y) - f(x, y)| \leq \varepsilon + B \sum_{i=0}^{3} |L_r(g_i; x, y) - g_i(x, y)|,
\]
where \( M := \varepsilon + N + \frac{4N}{\delta^2} \). Now for a given \( \rho > 0 \), choose \( \varepsilon > 0 \) such that \( \varepsilon < \rho \). Then, for each \( i = 0, 1, 2, 3 \), set \( \psi_\rho(x, y) := \{|k \in \mathbb{N} : |L_k(f; x, y) - f(x, y)| \geq \rho\} \) and \( \psi_{\rho, i}(x, y) := \{|k \in \mathbb{N} : |L_k(g_i; x, y) - g_i(x, y)| \geq \frac{\rho}{\delta_i^2} \} \) for \((i = 0, 1, 2, 3)\), it follows from (2.5) that \( \psi_{\rho, i}(x, y) \subseteq \bigcup_{i=0}^{3} \psi_{\rho, i}(x, y) \). Hence
\[
(2.6) \quad \frac{\|\psi_\rho(x, y)\|_{C_\beta(K)}}{h_r} \leq \sum_{i=0}^{3} \left( \frac{\|\psi_{\rho, i}(x, y)\|_{C_\beta(K)}}{h_r} \right).
\]
Now using hypothesis (2.2) and Definition 1.1 the right-hand side of (2.6) tends to zero as \( r \to \infty \). Therefore, we have \( \lim_{r \to \infty} \frac{1}{h_r} \|\psi_\rho(x, y)\|_{C_\beta(K)} = 0 \) for every \( \rho > 0 \), i.e., (2.1) holds.

Example 2.1. Consider the following Meyer-König and Zeller (2.13) (of two variables) operators:
\[
B_{m,n}(f; x, y) := (1 - x)^{m+1}(1 - y)^{n+1}
\]
easy to see that by Theorem 2.1, we have
\[ f_{\begin{array}{c} m, n \\ k, k + n + 1 \end{array}}(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{j}{m+j} \binom{k}{n+k} x^j y^k, \]
where \( f \in H_\omega(K) \), and \( K = [0, A] \times [0, B] \), \( A, B \in (0, 1) \).
Since, for \( x \in [0, A] \), \( A \in (0, 1) \), we have \( 1/(1-x)^{n+1} = \sum_{k=0}^{\infty} (n+k)x^k \), it is easy to see that \( B_{m,n}(g_0; x, y) = f_0(x, y) \). Also, we obtain
\[ B_{m,n}(g_1; x, y) = (1-x)^{n+1}(1-y)^{n+1} x \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{j}{m+1} \binom{m+j}{j} \binom{n+k}{k} x^j y^k \]
\[ = (1-x)^{n+1}(1-y)^{n+1} x \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{m+1} \binom{m+j}{j} \binom{n+k}{k} x^{j-1} y^k \]
and similarly \( B_{m,n}(g_2; x, y) = \frac{y}{1-y} \).
Finally, we get
\[ B_{m,n}(g_3; x, y) = (1-x)^{n+1}(1-y)^{n+1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{j}{m+1} \binom{k}{n+k} x^j y^k \]
\[ = (1-x)^{n+1}(1-y)^{n+1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{j}{m+1} \binom{m+j}{j} \binom{n+k}{k} x^j y^k \]
\[ + (1-x)^{n+1}(1-y)^{n+1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{m+1} \binom{m+j}{j} \binom{n+k}{k} x^{j-1} y^k \]
\[ = (1-x)^{n+1}(1-y)^{n+1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{j}{m+1} + \frac{1}{m+1} \right) \binom{m+j}{j} \binom{n+k}{k} x^j y^k \]
\[ + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{j}{m+1} + \frac{1}{m+1} \right) \binom{m+j}{j} \binom{n+k}{k} x^{j-1} y^k \]
\[ = \frac{m+2}{m+1} \left( \frac{x}{1-x} \right)^2 + \frac{1}{m+1} \frac{x}{1-x} + \frac{n+2}{n+1} \left( \frac{y}{1-y} \right)^2 + \frac{1}{n+1} \frac{y}{1-y} \]
\[ \to \left( \frac{x}{1-x} \right)^2 + \left( \frac{y}{1-y} \right)^2. \]
Therefore, we have \( B_n(f_3; x, y) \rightarrow g_3(x, y) \) (\( \theta \)-equistat) \( i = 0, 1, 2, 3 \). Hence by Theorem 2.1 we have \( B_n(f; x, y) \rightarrow g(x, y) \) (\( \theta \)-equistat).
3. Rate of Lacunary Equistatistical Convergence

In this section we study the rate of lacunary equistatistical convergence of a sequence of positive linear operators as given in [11].

**Definition 3.1.** Let \((a_n)\) be a positive non-increasing sequence. A sequence \((f_r)\) is said to be lacunary equistatistically convergent to a function \(f\) with the rate \(\beta\) \((0 < \beta < 1)\) if for every \(\epsilon > 0\),

\[
\lim_{r \to \infty} \frac{\Lambda_r(x, y)}{r^{-\beta}} = 0
\]

uniformly with respect to \((x, y) \in K\) or equivalently, for every \(\epsilon > 0\),

\[
\lim_{r \to \infty} \frac{\|\Lambda_r(x, y)\|_{C_0(X)}}{r^{-\beta}} = 0,
\]

where

\[
\Lambda_r(x, \epsilon) := \frac{1}{h_r} |\{k \in I_r : |f_k(x, y) - f(x, y)| \geq \epsilon\}| = 0.
\]

In this case, we write \(f_r \to f = o(r^{-\beta})\) \((\theta\text{-equistat})\) on \(K\).

We have the following basic lemma.

**Lemma 3.1.** Let \((f_r)\) and \((g_r)\) be sequences of functions belonging to \(H_\omega(K)\). Assume that \(f_r \to f = o(r^{-\beta_1})\) \((\theta\text{-equistat})\) on \(X\) and \(g_r \to g = o(r^{-\beta_2})\) \((\theta\text{-equistat})\).

Let \(\beta = \min\{\beta_1, \beta_2\}\). Then the following statement holds:

(i) \((f_r + g_r) - (f + g) = o(r^{-\beta})\) \((\theta\text{-equistat})\),

(ii) \((f_r - f)(g_r - g) = o(r^{-\beta_1})\) \((\theta\text{-equistat})\),

(iii) \(\mu(f_r - f) = o(r^{-\beta_1})\) \((\theta\text{-equistat})\) for any real number \(\mu\),

(iv) \(\sqrt{|f_r - f|} = o(r^{-\beta_1})\) \((\theta\text{-equistat})\).

We recall that the modulus of continuity of a function \(f \in H_\omega(K)\) is defined by

\[
\omega(f; \delta) = \sup_{s, x \in K} \{|f(s) - f(x)| : |s - x| \leq \delta\} \quad (\delta > 0).
\]

Now we prove the following result.

**Theorem 3.1.** Let \(\{L_r\}\) be a sequence of positive linear operators from \(H_\omega(K)\) into \(C_0(K)\). Assume that the following conditions hold:

(a) \(L_r(g_0; x, y) - g_0 = o(r^{-\beta_1})\) \((\theta\text{-equistat})\) on \(K\),

(b) \(\omega(f; \delta_{r,x}, \delta_{r,y}) = o(r^{-\beta_2})\) \((\theta\text{-equistat})\) on \(K\),

where \(\delta_{r,x} > \sqrt{L_r\left(\left((\frac{x}{1-r} - \frac{x}{1-r})^2, x\right)\right)}\) and \(\delta_{r,y} > \sqrt{L_r\left(\left((\frac{t}{1-r} - \frac{y}{1-r})^2, y\right)\right)}\). Then for all \(f \in H_\omega(K)\), we have \(L_r(f; x, y) - f(x, y) = o(r^{-\beta})\) \((\theta\text{-equistat})\) on \(K\), where \(\beta = \min\{\beta_1, \beta_2\}\).

**Proof.** Let \(f \in H_\omega(K)\) and \((x, y) \in K\). Then it is well known that,

\[
|L_r(f; x, y) - f(x, y)| \leq M |L_r(g_0; x, y) - g_0(x, y)|
+ (L_r(g_0; x, y) + \sqrt{L_r(g_0; x, y)}) \omega(f; \delta_{r,x}, \delta_{r,y}),
\]

where \(M\) is a modulus of continuity.
where $M = ||f||_{H,\omega(K)}$. This yields that
\[
|L_r(f; x, y) - f(x, y)| \leq M(||L_r(g_0; x, y) - g_0(x, y)|| + 2\omega(f; \delta_{r,x}, \delta_{r,y}) + \omega(f; \delta_{r,x}, \delta_{r,y}) (||L_r(g_0; x, y) - g_0(x, y)||).
\]
Now using the conditions (a), (b) and Lemma 3.1 in the above inequality, we get $L_r(f) - f = o(r^{-\beta})$ ($\theta$-equistat) on $K$.

References