TOPOLOGICALLY BOOLEAN AND \(g(x)\)-CLEAN RINGS

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Abstract. Let \(R\) be a ring with identity and let \(g(x)\) be a polynomial in \(Z(R)[x]\) where \(Z(R)\) denotes the center of \(R\). An element \(r \in R\) is called \(g(x)\)-clean if \(r = u + s\) for some \(u, s \in R\) such that \(u\) is a unit and \(g(s) = 0\). The ring \(R\) is \(g(x)\)-clean if every element of \(R\) is \(g(x)\)-clean. We consider \(g(x) = x(x - c)\) where \(c\) is a unit in \(R\) such that every root of \(g(x)\) is central in \(R\). We show, via set-theoretic topology, that among conditions equivalent to \(R\) being \(g(x)\)-clean, is that \(R\) is right (left) \(c\)-topologically boolean.

1. Introduction

Let \(R\) be a ring with identity and let \(g(x)\) be a polynomial in \(Z(R)[x]\) where \(Z(R)\) denotes the center of \(R\). Let \(\text{Id}(R)\) and \(U(R)\) denote the set of idempotents and the set of units in \(R\), respectively. The notion of \(g(x)\)-clean rings first appeared in a 2002 paper of Camillo and Simón [1], where an element \(r \in R\) is called \(g(x)\)-clean if \(r = u + s\) for some \(u \in U(R)\) and \(s \in R\) such that \(g(s) = 0\). The ring \(R\) is \(g(x)\)-clean if every element of \(R\) is \(g(x)\)-clean. Note that if \(r \in R\) is \(g(x)\)-clean and \(g(x)\) is a factor of a polynomial \(h(x) \in Z(R)[x]\), then \(r\) is also \(h(x)\)-clean.

Clearly, if \(g(x) = x^2 - x\), then \(g(x)\)-clean rings are clean. However, in general, \(g(x)\)-clean rings are not necessarily clean. A well-known example is the group ring \(\mathbb{Z}_7C_3\) where \(\mathbb{Z}_7 = \{m/n \mid m, n \in \mathbb{Z}, \gcd(7, n) = 1\}\) and \(C_3\) is the cyclic group of order 3. By [7] Example 2.7], \(\mathbb{Z}_7C_3\) is \((x^3 - 1)\)-clean. However, Han and Nicholson [4] have shown that \(\mathbb{Z}_7C_3\) is not clean.

Conversely, for a clean ring \(R\), there may exist a \(g(x) \in Z(R)[x]\) such that \(R\) is not \(g(x)\)-clean (see [3] Example 2.3]). Indeed, let \(R\) be a Boolean ring containing more than two elements. Let \(c \in R\) where \(0 \neq c \neq 1\) and let \(g(x) = x^2 + (1+c)x + c = (x + 1)(x + c)\). Since \(R\) is Boolean, so it is clean. Suppose that \(R\) is \(g(x)\)-clean. Then \(c = u + s\) for some \(u \in U(R)\) and \(s \in R\) such that \(g(s) = 0\). Note that \(u = 1\) since \(R\) is Boolean. Therefore, \(s = c + 1\). However, \(g(c + 1) = c \neq 0\) which contradicts the assumption that \(g(s) = 0\). Hence, it follows that \(R\) is clean but not \(g(x)\)-clean.

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In [2], a ring \( R \) (not necessarily commutative) is said to be right (left) topologically boolean, or a right (left) tb-ring for short, if for every pair of distinct maximal right (left) ideals of \( R \), there is a nontrivial idempotent in exactly one of them. The case where \( R \) is commutative has been considered earlier in [5]. Now let \( g_c(x) = x(x-c) \in \mathbb{Z}[R][x] \). Here, we define a ring \( R \) to be right (left) \( c \)-topologically boolean, or a right (left) \( c \)-tb ring for short, if for every pair of distinct maximal right (left) ideals of \( R \), there is a root of \( g_c(x) \) in exactly one of them. We say that \( R \) is a \( c \)-tb ring if it is both right and left \( c \)-tb. Clearly, when \( c = 1 \), a right (left) \( c \)-tb ring is just a right (left) tb-ring.

In this paper we consider \( g(x) = x(x-c) \in \mathbb{Z}[R][x] \) where \( c \) is a unit in \( R \) such that every root of \( g(x) \) is central in \( R \) and show via set-theoretic topology that among conditions equivalent to \( R \) being \( g(x) \)-clean is that \( R \) is right (left) \( c \)-tb. Throughout this paper, all rings are assumed to be associative with identity.

2. Some preliminaries

Let \( n \) be a positive integer. For a ring \( R \) and polynomial \( g(x) \in \mathbb{Z}[R][x] \), an element \( r \in R \) is said to be \( (n, g(x)) \)-clean if \( r = u_1 + \cdots + u_n + s \) for some \( u_1, \ldots, u_n \in U(R) \) and \( s \in R \) such that \( g(s) = 0 \). The ring \( R \) is \( (n, g(x)) \)-clean if all of its elements are \( (n, g(x)) \)-clean. Clearly, a \( (1, g(x)) \)-clean ring is \( g(x) \)-clean. In [3], an element \( r \in R \) is said to be \( n \)-clean if \( r = e + u_1 + \cdots + u_n \) for some \( e \in \text{Id}(R) \) and \( u_1, \ldots, u_n \in U(R) \). The ring \( R \) is \( n \)-clean if all of its elements are \( n \)-clean.

In [7] Theorem 2.1, Wang and Chen showed that if \( g(x) = (x-a)(x-b) \in \mathbb{Z}[R][x] \) with \( b - a \in U(R) \), then \( R \) is \( g(x) \)-clean if and only if \( R \) is clean. In [3] Theorem 3.2, Fan and Yang gave another proof of the same result. In the following, we give an extension to \( n \)-clean rings as follows:

**Theorem 2.1.** Let \( R \) be a ring and let \( g(x) = (x-a)(x-b)h(x) \in \mathbb{Z}[R][x] \) such that \( b - a \in U(R) \). If \( R \) is \( n \)-clean, then \( R \) is \( (n, g(x)) \)-clean \((n \in \mathbb{N})\).

**Proof.** Let \( r \in R \). Since \( R \) is \( n \)-clean, then \( (r-a)(b-a)^{-1} = e + u_1 + \cdots + u_n \) for some \( e \in \text{Id}(R) \) and \( u_i \in U(R) \) \((i = 1, \ldots, n)\). Thus, \( r = (e(b-a) + u_1(b-a) + \cdots + u_n(b-a), \text{ where } u_i(b-a) \in U(R) \) \((i = 1, \ldots, n)\). Note that \( g(e(b-a) + u_1(b-a) + \cdots + u_n(b-a)) = 0 \). Hence, \( e(b-a) + u_1(b-a) + \cdots + u_n(b-a) \) is a root of \( g(x) \). It follows that \( R \) is \((n, g(x))\)-clean. \(\Box\)

By Theorem 2.1 and the fact that clean rings are \( n \)-clean for any integer \( n \geq 1 \) (by [9] Lemma 2.1), we obtain the following:

**Corollary 2.1.** Let \( R \) be a ring and let \( g(x) = (x-a)(x-b) \in \mathbb{Z}[R][x] \) such that \( b - a \in U(R) \). Then \( R \) is \((g(x))\)-clean if and only if \( R \) is \((n, g(x))\)-clean for all positive integers \( n \).

Let \( R \) be a ring and let \( g(x) \in \mathbb{Z}[R][x] \). An element \( r \in R \) is called weakly \( g(x) \)-clean if \( r = u + s \) or \( r = u - s \) for some \( u \in U(R) \) and \( s \in R \) such that \( g(s) = 0 \). We say that \( R \) is weakly \( g(x) \)-clean if every element in \( R \) is weakly \( g(x) \)-clean. Clearly, a \( g(x) \)-clean ring is weakly \( g(x) \)-clean. It is also clear that if
$R$ is a weakly $g(x)$-clean ring and $g(x)$ is a factor of a polynomial $h(x) \in Z(R)[x]$, then $R$ is also a weakly $h(x)$-clean ring.

In the following we obtain some results which generalise parts of Theorem 3.5 in [3].

**Proposition 2.1.** Let $R$ be a ring which is weakly $x(x - c)$-clean where $c \in Z(R)$. Then $c \in U(R)$.

**Proof.** Let $g(x) = x(x - c) \in Z(R)[x]$. Since $R$ is weakly $g(x)$-clean, $c = u + s$ or $c = u - s$ for some $u \in U(R)$ and $s \in R$ such that $g(s) = 0$. For the case $c = u + s$, we have that $s = -u + c$ and hence, $s^2 = (-u + c)^2 = u^2 + cr$ for some $r \in R$. Since $0 = g(s) = s(s - c)$, we also have $s^2 = cs$. Thus, $c(s - r) = u^2 \in U(R)$. This implies that $c \in U(R)$. For the case $c = u - s$, we have that $s = u - c$ and hence, $s^2 = (u - c)^2 = u^2 - cr$ for some $r \in R$. Since $0 = g(s) = s(s - c)$, we also have $s^2 = cs$. Thus, $c(s + r) = u^2 \in U(R)$ which implies that $c \in U(R)$. □

**Lemma 2.1.** Let $R$ be a ring and let $g(x) = ax^m - bx^n$, $h(x) = ax^m + bx^n \in Z(R)[x]$ where $m, n$ are positive integers of different parity. Then $R$ is $g(x)$-clean if and only if $R$ is $h(x)$-clean.

**Proof.** ($\Rightarrow$): Assume that $R$ is a $g(x)$-clean ring. Then for any $r \in R$, $-r = u + s$ where $u \in U(R)$ and $s \in R$ such that $g(s) = 0$. It follows that $r = (-u) + (-s)$. Note that

$$h(-s) = a(-s)^m + b(-s)^n = (-1)^m as^m + (-1)^n bs^n$$

$$= \begin{cases} as^m - bs^n, & \text{if } m \text{ is even, } n \text{ is odd} \\ -(as^m - bs^n), & \text{if } m \text{ is odd, } n \text{ is even} \end{cases}$$

$$= 0.$$

It follows that $r$ is $h(x)$-clean.

($\Leftarrow$): Suppose that $R$ is $h(x)$-clean. Then for any $r \in R$, $-r = u + s$ where $u \in U(R)$ and $s \in R$ such that $h(s) = 0$. It follows that $r = (-u) + (-s)$. Then since

$$g(-s) = a(-s)^m - b(-s)^n = (-1)^m as^m - (-1)^n bs^n$$

$$= \begin{cases} as^m + bs^n, & \text{if } m \text{ is even, } n \text{ is odd} \\ -(as^m + bs^n), & \text{if } m \text{ is odd, } n \text{ is even} \end{cases}$$

$$= 0,$$

we have that $r$ is $g(x)$-clean. □

**Theorem 2.2.** Let $R$ be a ring and let $c \in Z(R)$. Then the following are equivalent:

(a) $R$ is $x(x - c)$-clean;
(b) $R$ is $x(x + c)$-clean;
(c) $R$ is $n$-clean for all positive integers $n$ and $c \in U(R)$.
Proof. (a)⇒ (b): This follows readily by Lemma 2.1. (a)⇒ (c): Assume (a). By Proposition 2.1, we have \( c \in U(R) \). It follows by Corollary 2.1 that \( R \) is \( n \)-clean for all positive integers \( n \).

(c)⇒ (a): This follows readily by Theorem 2.1 (take \( n = 1 \)).

Lemma 2.2. Let \( R \) be a ring, let \( c \in U(R) \) and let all roots of \( g(x) = x(x - c) \) in \( R \) be central. For any \( a, b \in R \), if \( ab = c \), then \( ba = c \).

Proof. Let \( a, b \in R \) such that \( ab = c \). Since \( c \) is a root of \( g(x) \), we have that \( c \) is central and therefore, \( ba(ba - c) = baba - c(ba) = b(ab)a - c(ba) = c(ba) - c(ba) = 0 \).

Thus, \( ba \) is a root of \( g(x) \) and hence, \( ba \) is also central. Then \( ca = (ab)a = a(ba) = baa \) and it follows that \( c^2 = c(ab) = (ca)b = (ba)a = bac \). Since \( c \in U(R) \) (by the hypothesis), it follows that \( c = ba \).

3. Some equivalent conditions for \( x(x - c) \)-clean rings

Let \( R \) be a ring. A proper right (left) ideal \( P \) of \( R \) is said to be prime if \( aRb \subseteq P \) with \( a, b \in R \) implies that \( a \in P \) or \( b \in P \). Given a ring \( R \), let \( \text{Spec}_r(R) \) be the set of all maximal right ideals of \( R \) which are prime. It has been shown in [10] Corollary 2.8 that if \( R \) is not a right quasi-duo ring, then \( \text{Spec}_r(R) \) is a topological space with the weak Zariski topology but not with the Zariski topology. For a right ideal \( I \) of \( R \), let \( \mathcal{U}_r(I) = \{ P \in \text{Spec}_r(R) \mid P \nsubseteq I \} \) and \( \mathcal{V}_r(I) = \text{Spec}_r(R) \setminus \mathcal{U}_r(I) \). Let \( \tau = \{ \mathcal{U}_r(I) \mid I \text{ is a right ideal of } R \} \). Then \( \tau \) contains the empty set and \( \text{Spec}_r(R) \). In general, \( \tau \) is just a subspace of the weak Zariski topology on \( \text{Spec}_r(R) \).

For any element \( a \in R \), let \( \mathcal{U}_r(a) = \mathcal{U}_r(aR) \) and \( \mathcal{V}_r(a) = \mathcal{V}_r(aR) \). Then \( \mathcal{U}_r(a) = \{ P \in \text{Spec}_r(R) \mid a \notin P \} \) and \( \mathcal{V}_r(a) = \{ P \in \text{Spec}_r(R) \mid a \in P \} \). The left prime spectrum \( \text{Spec}_l(R) \) and the weak Zariski topology associated with it are defined analogously. Let \( \text{Max}_r(R) = (\text{Max}_l(R)) \) be the set of all maximal right (left) ideals of \( R \). Since maximal right (left) ideals are prime right (left) ideals (see [6], \( \text{Max}_r(R) = (\text{Max}_l(R)) \) inherits the weak Zariski topology on \( \text{Spec}_r(R) \) (\( \text{Spec}_l(R) \)).

Let \( U_r(I) = \text{Max}_r(R) \cap \mathcal{U}_r(I) \) and \( V_r(I) = \text{Max}_r(R) \cap \mathcal{V}_r(I) \) for any right ideal \( I \) of \( R \). Then, in particular, \( U_r(a) = \text{Max}_r(R) \cap \mathcal{U}_r(a) \) and \( V_r(a) = \text{Max}_r(R) \cap \mathcal{V}_r(a) \) for any \( a \in R \).

Recall that a clopen set in a topological space is a set which is both open and closed. A topological space is said to be zero-dimensional if it has a base consisting of clopen sets.

We begin with the following lemmas.

Lemma 3.1. Let \( R \) be a ring, let \( g(x) = x(x - c) \in Z(R)[x] \) where \( c \in U(R) \) and let \( s \in R \) be a central root of \( g(x) \). Let \( N \) be a maximal right ideal of \( R \). If \( s \notin N \), then \( c - s \in N \).

Proof. Since \( g(s) = 0 \), we have that \( s(s - c) = 0 \in P \) for any prime right ideal \( P \) of \( R \). Then since \( s \) is central, it follows that every prime right ideal of \( R \) contains either \( s \) or \( s - c \). Now since \( c = s + (c - s) \) and \( c \in U(R) \), we have that \( 1 = sc^{-1} + (c - s)c^{-1} \). Hence, every prime right ideal of \( R \) contains either \( s \) or \( c - s \) but not both. Since maximal right ideals are prime right ideals (by [6]), it follows that if \( s \notin N \), then \( c - s \in N \).
Let $R$ be a ring and let $g(x) = x(x - c) \in Z(R)[x]$ where $c \in U(R)$. Let $s, t \in R$ be central roots of $g(x)$. Then $c^{-1}st, s + t - c^{-1}st$ and $c - s$ are also roots of $g(x)$.

**Proof.** We first note that since $s(s - c) = 0$ and $t(t - c) = 0$, we thus have $s = c^{-1}s^2$ and $t = c^{-1}t^2$. Then

$$g(c^{-1}st) = c^{-1}st(c^{-1}st - c) = c^{-2}(st)^2 - st$$

$$= c^{-2}(st)^2 - c^{-1}s^2t = c^{-2}(s^2t)(t - c) = 0.$$ We also have that

$$g(s + t - c^{-1}st) = (s + t - c^{-1}st)(s + t - c^{-1}st - c)$$

$$= s(s - c) + s(t - c^{-1}st) + (s - c^{-1}st)$$

$$+ t(t - c^{-1}st(s - c^{-1}st) - c^{-1}st(t - c) = 0.$$ Finally, we note that $g(c - s) = (c - s)((c - s) - c) = (s - c)s = g(s) = 0$. 

Let $R$ be a ring and let $g(x) = x(x - c) \in Z(R)[x]$ where $c \in U(R)$. Let $\xi = \{U_r(s) \mid s \in R \text{ is a central root of } g(x) = x(x - c)\}$. By Lemma 3.2 and the following lemma, we may deduce that $\xi$ is closed under intersection and union.

**Lemma 3.3.** Let $R$ be a ring and let $g(x) = x(x - c) \in Z(R)[x]$ with $c \in U(R)$ such that every root of $g(x)$ is central in $R$. If $s, t \in R$ are roots of $g(x)$, then the following hold.

(a) $U_r(s) \cap U_r(t) = U_r(c^{-1}st)$;
(b) $U_r(s) \cup U_r(t) = U_r(s + t - c^{-1}st)$;
(c) $U_r(s) = V_r(c - s)$. In particular, every set in $\xi$ is clopen.

**Proof.** (a) Let $P \in U_r(s) \cap U_r(t)$. Then $P \in \text{Spec}_r(R)$ with $s, t \notin P$. Note that $c \notin P$. Since $c, s, t$ are central in $R$ and $P$ is a prime right ideal of $R$, it follows that $c^{-1}st \notin P$. Hence, $P \in U_r(c^{-1}st)$ and therefore, $U_r(s) \cap U_r(t) \subseteq U_r(c^{-1}st)$. Conversely, suppose that $P \in U_r(c^{-1}st)$. If $s$ or $t$ belongs to $P$, then since $s, t$ are central in $R$ and $P$ is a right ideal of $R$, it follows that $c^{-1}st \in P$; a contradiction. Thus $s$ and $t$ do not belong to $P$, that is, $P \in U_r(s) \cap U_r(t)$. Hence, $U_r(c^{-1}st) \subseteq U_r(s) \cap U_r(t)$. The equality $U_r(s) \cap U_r(t) = U_r(c^{-1}st)$ thus follows. Then $U_r(s) \cap U_r(t) = U_r(s) \cap U_r(t) \cap \text{Max}_r(R) = U_r(c^{-1}st) \cap \text{Max}_r(R) = U_r(c^{-1}st)$.

(b) Let $P \in U_r(s) \cup U_r(t)$. Then $s \notin P$ or $t \notin P$. Without loss of generality, suppose that $s \notin P$. Since $s(s - c) = 0 \in P$ and $s \notin P$ with $s$ central in $R$, it follows that $s - c \in P$. Then $(1 - c^{-1}s)t = -c^{-1}(s - c)t \in P$. If $s + (1 - c^{-1}s)t \in P$, then it will follow that $s \in P$; a contradiction. Thus, $s + (1 - c^{-1}s)t \notin P$ and hence, $P \in U_r(s + (1 - c^{-1}s)t)$. The inclusion $U_r(s) \cap U_r(t) \subseteq U_r(s + (1 - c^{-1}s)t)$ therefore holds. For the reverse inclusion, suppose that $P \in U_r(s + (1 - c^{-1}s)t)$. Then $s + (1 - c^{-1}s)t \notin P$. If $s$ and $t$ both belong to $P$, then $s + (1 - c^{-1}s)t \in P$; a contradiction. Hence, either $s \notin P$ or $t \notin P$, that is, $P \in U_r(s)$ or $P \in U_r(t)$. Therefore, $P \in U_r(s) \cup U_r(t)$ and the inclusion $U_r(s + (1 - c^{-1}s)t) \subseteq U_r(s) \cup U_r(t)$
follows. Hence, \( U_r(s) \cup U_r(t) = U_r(s + (1 - c^{-1})t) \). It follows that
\[
U_r(s) \cup U_r(t) = (U_r(s) \cap \text{Max}_r(R)) \cup (U_r(t) \cap \text{Max}_r(R))
\]
\[
= (U_r(s) \cup U_r(t)) \cap \text{Max}_r(R)
\]
\[
= U_r(s + (1 - c^{-1})t) \cap \text{Max}_r(R) = U_r(s + (1 - c^{-1})t).
\]

(c) By using Lemma 3.1, we have \( U_r(s) = \text{Max}_r(R) - U_r(c - s) = V_r(c - s) \). It follows that every set in \( \xi \) is clopen.

Next, we extend Proposition 2.4 in \cite{2} as follows:

**Proposition 3.1.** Let \( R \) be an \( x(x - c) \)-clean ring with \( c \in Z(R) \) such that every root of \( x(x - c) \) is central in \( R \). Then \( R \) is a right \( c \)-tb ring.

**Proof.** By Proposition 2.4, \( c \in U(R) \). Let \( M \) and \( N \) be distinct maximal right ideals of \( R \). Then there exists \( a \in M \setminus N \) and \( N + aR = R \). Hence, \( 1 - ar \in N \) for some \( r \in R \). Since \( N \) is a right ideal of \( R \), \( c - arc = (1 - ar)c \in N \). Let \( y = arc \). Then \( c - y \in N \) and \( y \in M \setminus N \). Since \( R \) is \( x(x - c) \)-clean, there exist a unit \( u \in R \) and a root \( s \in R \) of \( x(x - c) \) such that \( y = u + s \). If \( s \in M \), then \( u = y - s \in M \) from which it follows that \( M = R \); a contradiction since \( M \) is a maximal right ideal of \( R \). Thus, \( s \notin M \). If \( s \notin N \), then \( c - s \in N \) (by Lemma 3.1) and hence, \( u = y - s = (y - c) + (c - s) \in N \). It follows that \( N = R \) which is also not possible since \( N \) is a maximal right ideal of \( R \). We thus have that \( s \) is a root of \( x(x - c) \) belonging to \( N \) only. Hence, \( R \) is a right \( c \)-tb ring.

**Proposition 3.2.** Let \( R \) be a ring and let \( g(x) = x(x - c) \in Z(R)[x] \) with \( c \in U(R) \) such that every root of \( g(x) \) in \( R \) is central. If \( R \) is a right \( c \)-tb ring, then \( \xi \) forms a base for the weak Zariski topology on \( \text{Max}_r(R) \). In particular, \( \text{Max}_r(R) \) is a compact, zero-dimensional Hausdorff space.

**Proof.** Note that if \( M_1 \) and \( M_2 \) are two distinct maximal right ideals of \( R \), then \( R \) is a right \( c \)-tb ring, there exists a root \( s \in R \) of \( g(x) \) such that \( s \notin M_1 \), \( s \in M_2 \) (that is, \( M_1 \in U_r(s), M_2 \notin U_r(s) \)). The points in \( \text{Max}_r(R) \) can therefore be separated by disjoint clopen sets belonging to \( \xi \). Hence, \( \text{Max}_r(R) \) is Hausdorff.

To show that \( \xi \) forms a base for the weak Zariski topology on \( \text{Max}_r(R) \), let \( K \subseteq \text{Max}_r(R) \) be a closed subset and take \( M \notin K \). For each \( N \in K \), since \( N \neq M \), there exists a clopen set \( U_r(s_N) \in \xi \) separating \( M \) and \( N \), say \( N \in U_r(s_N) \). The collection \( \{U_r(s_N) \mid N \in K \} \) is therefore an open cover of the set \( K \). Since \( K \) is compact, it has a finite subcover, that is, \( K \) is contained in a finite cover of sets of the form \( U_r(s_N) \) with \( N \in K \). By Lemma 3.3, there exists a clopen set \( C \in \xi \) separating \( M \) from \( K \). Hence, \( \xi \) forms a base for the weak Zariski topology on \( \text{Max}_r(R) \). Since every set in \( \xi \) is clopen (by Lemma 3.3), it follows that \( \text{Max}_r(R) \) is zero-dimensional.

**Proposition 3.3.** Let \( R \) be a ring and let \( g(x) = x(x - c) \in Z(R)[x] \) with \( c \in U(R) \) such that every root of \( g(x) \) in \( R \) is central. If \( \xi \) forms a base for the weak Zariski topology on \( \text{Max}_r(R) \), then for any \( a \in R \), there exists a root \( s \) of \( g(x) \) such that \( s \notin M \) for every \( M \in V_r(a) \) and \( s \in N \) for every \( N \in V_r(a - c) \).
Proof. Consider the disjoint closed sets $V_r(a)$ and $V_r(a - c)$. Since $\xi$ forms a base for the weak Zariski topology on $\text{Max}_1(R)$ and $\text{Max}_1(R)$ is compact, there is a clopen set $U_r(s) \in \xi$ separating the sets $V_r(a)$ and $V_r(a - c)$. Without loss of generality, assume that $V_r(a) \subseteq U_r(s)$ and $V_r(a - c) \subseteq V_r(s)$. Then it follows that $s \notin M$ for every $M \in V_r(a)$ and $s \in N$ for every $N \in V_r(a - c)$.

Proposition 3.1. Let $R$ be a ring and let $g(x) = x(x - c) \in Z(R)[x]$ with $c \in U(R)$ such that every root of $g(x)$ in $R$ is central. If for every $a \in R$ there exists a root $s \in Z(R)$ of $g(x)$ such that $V_r(a) \subseteq U_r(s)$ and $V_r(a - c) \subseteq V_r(s)$, then $R$ is $g(x)$-clean.

Proof. Let $a \in R$. By the hypothesis, there exists a root $s \in Z(R)$ of $g(x)$ such that $V_r(a) \subseteq U_r(s)$ and $V_r(a - c) \subseteq V_r(s)$. We claim that $a - s$ is a unit. Let $M$ be a maximal right ideal of $R$. Note that if $a \in M$, then $a - s \notin M$, since $s \notin M$. Next, suppose that $a \notin M$. If $a - s \in M$, then $s \notin M$, and hence, $c - s \in M$ (by Lemma 3.1). Then since $(a - c) + (c - s) = a - s \in M$, it follows that $a - c \in M$ and hence, $s \in M$ (because $V_r(a - c) \subseteq V_r(s)$); a contradiction. Thus, $a - s \notin M$. We have therefore shown that $a - s \notin M$ for any maximal right ideal $M$ of $R$. Hence, $a - s$ has a right inverse, that is, $(a - s)v = 1$ for some $v \in R$. Then $(a - s)(ve) = c$ and by Lemma 2.2, we have that $(ve)(a - s) = c$. Since $c \in U(R) \cap Z(R)$, we can conclude that $a - s$ is a unit in $R$. Hence, $a$ is the sum of a unit and a root of $g(x)$ in $R$. Since $a$ is arbitrary in $R$, it follows that $R$ is $g(x)$-clean.

We are now ready for the main result.

Theorem 3.1. Let $R$ be a ring and let $x(x - c) \in Z(R)[x]$ with $c \in U(R)$. If every root of $x(x - c)$ is central in $R$, then the following conditions are equivalent.

(a) $R$ is $x(x - c)$-clean;
(b) $R$ is $x(x + c)$-clean;
(c) $R$ is $n$-clean for all positive integers $n$;
(d) $R$ is a right $c$-tb ring;
(e) The collection $\xi = \{U_r(s) \mid s \in R$ is a root of $x(x - c)\}$ forms a base for the weak Zariski topology on $\text{Max}_1(R)$;
(f) For every $a \in R$, there exists a root $s \in Z(R)$ of $x(x - c)$ such that $V_r(a) \subseteq U_r(s)$ and $V_r(a - c) \subseteq V_r(s)$;
(g) $R$ is a left $c$-tb ring;
(h) The collection $\xi = \{U_l(s) \mid s \in R$ is a root of $x(x - c)\}$ forms a base for the weak Zariski topology on $\text{Max}_1(R)$.

Proof. By Theorem 2.2, it follows readily that (a) $\iff$ (b) $\iff$ (c). By Proposition 3.1, we readily have (a) $\Rightarrow$ (d). The implications (d) $\Rightarrow$ (e) $\Rightarrow$ (f) follow by Propositions 3.2 and 3.3 respectively. The implication (f) $\Rightarrow$ (a) is straightforward by using Proposition 3.4. By using the left analogue of the arguments in the proofs of (a) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (a), we obtain the equivalence (a) $\iff$ (g) $\iff$ (h).

A ring $R$ is said to be strongly clean if every element of $R$ is the sum of an idempotent and a unit which commute with one another. A strongly clean ring is...
therefore clean and hence, \(x(x-1)\)-clean. On the other hand, an abelian \(x(x-1)\)-clean ring is clearly strongly clean. We thus have the following as a consequence of Theorem 3.1:

**Corollary 3.1.** Let \(R\) be an abelian ring. The following conditions are equivalent:

(a) \(R\) is clean;
(b) \(R\) is strongly clean;
(c) \(R\) is \(x(x+1)\)-clean;
(d) \(R\) is \(n\)-clean for all positive integers \(n\);
(e) \(R\) is a right tb-ring;
(f) The collection \(\xi = \{U_r(s) \mid s \in \text{Id}(R)\}\)

forms a base for the weak Zariski topology on \(\text{Max}_r(R)\);
(g) For every \(a \in R\), there exists \(s \in \text{Id}(R)\)

such that \(V_r(a) \subseteq U_r(s)\) and \(V_r(a-1) \subseteq V_r(s)\);
(h) \(R\) is a left tb-ring;
(i) The collection \(\xi = \{U_l(s) \mid s \in \text{Id}(R)\}\)

forms a base for the weak Zariski topology on \(\text{Max}_l(R)\).

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