RINGS IN WHICH THE POWER OF EVERY ELEMENT IS THE SUM OF AN IDEMPOTENT AND A UNIT

Huanyin Chen and Marjan Sheibani

Abstract. A ring $R$ is uniquely $\pi$-clean if the power of every element can be uniquely written as the sum of an idempotent and a unit. We prove that a ring $R$ is uniquely $\pi$-clean if and only if for any $a \in R$, there exists an integer $m$ and a central idempotent $e \in R$ such that $a^m - e \in J(R)$, if and only if $R$ is Abelian; idempotents lift modulo $J(R)$; and $R/P$ is torsion for all prime ideals $P \supseteq J(R)$. Finally, we completely determine when a uniquely $\pi$-clean ring has nil Jacobson radical.

1. Introduction

An attractive problem in ring theory is to determine when a ring is generated additively by idempotents and units. An element of a ring is uniquely clean if it can be uniquely written as the sum of an idempotent and a unit. A ring $R$ is uniquely clean if every element in $R$ is uniquely clean. Many results on such rings can be found in [3, 5, 6]. Following Zhou [6], a ring $R$ is uniquely $\pi$-clean if some power of every element in $R$ is uniquely clean. This is a natural generalization of uniquely clean rings. The motivation of this paper is to develop explicit characterizations of such rings.

In Section 2, we explore the structures of uniquely $\pi$-clean rings, and prove that a ring $R$ is uniquely $\pi$-clean if and only if for any $a \in R$, there exists an $m \in \mathbb{N}$ and a central idempotent $e \in R$ such that $a^m - e \in J(R)$, if and only if for any $a \in R$, there exists an $m \in \mathbb{N}$ and a unique $e \in R$ such that $a^m - e \in J(R)$, and $J(R) = \{x \in R \mid x^m - 1 \in U(R)\}$ for all $m \in \mathbb{N}$. This extends Lee and Zhou’s theorem as well.

In Section 3, we characterize uniquely $\pi$-cleaness by means of certain prime ideals. It is shown that a ring $R$ is uniquely $\pi$-clean if and only if $R$ is Abelian; every idempotent lifts modulo $J(R)$; and $R/P$ is torsion for all prime ideals $P$ containing $J(R)$.

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exists a unique idempotent radical. Recall that an element \( a \in R \) is uniquely nil-clean provided that there exists a unique idempotent \( e \in R \) such that \( a - e \in N(R) \). We say that \( a \in R \) is uniquely \( \pi \)-nil-clean if for any right \( R \)-module \( A \) and any two decompositions \( A = M \oplus N = \bigoplus_{i \in I} A_i \), where \( M_R \cong R \) and the index set \( I \) is finite, there exist submodules \( A' \subseteq A_i \) such that \( A = M \oplus (\bigoplus_{i \in I} A'_i) \). The class of exchange rings is very large. For instance, regular rings, \( \pi \)-regular rings, strongly \( \pi \)-regular rings, semiperfect rings, left or right continuous rings, clean rings and unit \( C^* \)-algebras of real rank zero, etc. We begin with

**Lemma 2.1.** Every uniquely \( \pi \)-clean ring is an Abelian exchange ring.

**Proof.** Let \( R \) be uniquely \( \pi \)-clean, let \( e \in R \) be an idempotent, and let \( r \in R \). Then \( x := 1 - (e + er(1-e)) \in R \) is an idempotent. By hypothesis, \( x \in R \) is uniquely clean. One easily checks that

\[
x = e + (1 - 2e - er(1-e)) = (e + er(1-e)) + (1 - 2(e + er(1-e))).
\]

Further,

\[
e^2 = e^2 \in R, \quad (1 - 2e - er(1-e))^{-1} = (1 - er(1-e))(1 - 2e),
\]

\[
(e + er(1-e)) = (e + er(1-e))^2, \quad (1 - 2(e + er(1-e)))^2 = 1.
\]

By the uniqueness, we get \( e = e + er(1-e) \), and then \( er = ere \). Likewise, \( re = ere \). Thus, \( er = re \), and therefore \( R \) is Abelian.

For any \( a \in R \), then we can find some \( m \in \mathbb{N} \) such that \( a^m \in R \) is clean. Write \( a^m = f + v \), where \( f = f^2, v \in U(R) \). Then \( a^m - f^m = v \), and so \( a - f \in U(R) \).
This implies that \( R \) is strongly clean. In view of [9, Theorem 30.2], every clean ring is an exchange ring. Therefore \( R \) is an exchange ring, as asserted.

A ring \( R \) is strongly clean if for any \( a \in R \) there exists an idempotent \( e \in R \) such that \( a - e \in U(R) \) and \( ea = ae \). As a consequence of Lemma 2.1, every uniquely \( \pi \)-clean ring is strongly clean. A ring \( R \) is uniquely clean provided that every element in \( R \) can be uniquely written as the sum of an idempotent and a unit. It is easy to verify that \( \mathbb{Z}/3\mathbb{Z} \) is not uniquely clean as \( 2 = 0 + 2 = 1 + 1 \), while \( \mathbb{Z}/3\mathbb{Z} \) is uniquely \( \pi \)-clean. Let \( R = \bigoplus \mathbb{Z}_p \) is prime \( \mathbb{Z}/(p+1)\mathbb{Z} \). Then \( R \) is strongly clean. For any \( 1 \leq m \leq \lceil \log_2 p \rceil \), \( 2^m \in \mathbb{Z}/(p+1)\mathbb{Z} \) is not uniquely clean. Thus, \( R \) is not uniquely \( \pi \)-clean. Therefore, we conclude that \{uniquely clean rings\} \( \subset \{ \) strongly clean rings \( \} \).

**Theorem 2.1.** Let \( R \) be a ring. Then \( R \) is uniquely \( \pi \)-clean if and only if

1. \( R \) is Abelian;
2. Every idempotent lifts modulo \( J(R) \);
3. \( R/J(R) \) is uniquely \( \pi \)-clean.

**Proof.** Suppose \( R \) is uniquely \( \pi \)-clean. In view of Lemma 2.1, \( R \) is an Abelian exchange ring. This proves (1) and (2), in terms of [9, Theorem 30.2]. For any \( a \in R/J(R) \), then \( a \in R \) is uniquely \( \pi \)-clean. Thus, we have some \( n \in \mathbb{N} \) such that \( a^n \in R \) is uniquely clean. This implies that \( a^n = e + u, e = e^2 \in R, u \in U(R) \). Hence, \( a^n = \overline{1} + \overline{u} \). Write \( \overline{a^n} = \overline{f} + \overline{v}, \overline{f} = \overline{f^2} \in R/J(R), \overline{v} \in U(R) \). Clearly, every unit lifts modulo \( J(R) \). So we may assume that \( f = f^2 \in R, v \in U(R) \). As a result, there exists some \( r \in J(R) \) such that \( a^n = e + u = f + (v + r) \). By the uniqueness, we get \( e = f \). Therefore \( R/J(R) \) is uniquely \( \pi \)-clean.

Conversely, assume that (1)–(3) hold. For any \( a \in R \), we have \( \overline{a} \in R/J(R) \), and so there exists some \( n \in \mathbb{N} \) such that \( \overline{a^n} \in R \) is uniquely clean. By hypothesis, idempotents lift modulo \( J(R) \). In addition, units lift modulo \( J(R) \). Thus, \( a^n = e + u, e = e^2 \in R, u \in U(R) \). Write \( a^n = f + v, f = f^2 \in U(R) \). Then \( a^n = \overline{f} + \overline{v} \). By the uniqueness, we get \( \overline{v} = \overline{f} \), i.e., \( e = f \in J(R) \). This infers that \( f(1 - e) = (e - f)(e - 1) \in J(R) \). As every idempotent in \( R \) is central, \( f(1 - e) \in R \) is an idempotent, thus, \( f(1 - e) = 0 \). It follows that \( f = fe \). Likewise, \( e = ef \). Consequently, \( e = f \), and therefore \( R \) is uniquely \( \pi \)-clean.

**Corollary 2.1.** Every corner of a uniquely \( \pi \)-clean ring is uniquely \( \pi \)-clean.

**Proof.** Let \( R \) be uniquely \( \pi \)-clean, and let \( e = e^2 \in R \). In light of Theorem 2.1, \( e \in R \) is central. For any \( eae \in eRe \), then \( e(a + 1 - e) \in R \) is uniquely \( \pi \)-clean. So we have some \( n \in \mathbb{N} \) such that \( (eae + 1 - e)^n \in R \) is uniquely clean. Thus, \( (eae + 1 - e)^n = f + u, f = f^2 \in R, u \in U(R) \), and so \( (eae)^n = efe + eue \) is clean in \( eRe \). Write \( (eae)^n = g + v, g = g^2 \in eRe, v \in U(eRe) \). Then \( (eae + 1 - e)^n = (eae)^n + 1 - e = g + (v + 1 - e) \), where \( g = g^2 \in R \). Write \( vw = wv = e \). Then \( (v + 1 - e)^n = w + 1 - e \), and so \( v + 1 - e \in U(R) \). Thus, \( g = f = ege = efe \), as required.
Lemma 2.1 shows that every uniquely \( \pi \)-clean ring is an Abelian exchange ring. We now exhibit an exchange-like property of such rings.

**Theorem 2.2.** Let \( R \) be a ring. Then \( R \) is uniquely \( \pi \)-clean if and only if

1. \( R \) is Abelian;
2. For any \( a \in R \), there exists an \( n \in \mathbb{N} \) and a unique idempotent \( e \in a^n R \) such that \( 1 - e \in (1 - a^n)R \).

**Proof.** Suppose that \( R \) is uniquely \( \pi \)-clean. In view of Lemma 2.1, every idempotent in \( R \) is central. For any \( a \in R \), there exists some \( n \in \mathbb{N} \) such that \( a^n \in R \) is uniquely clean. Write \( a^n = f + v \), where \( f = f^2 \), \( v \in U(R) \). Set \( g = 1 - f \). Then \( g = g^2 \in R \). Obviously, we get

\[(a^n - g)v = (f + v - v(1 - f)v^{-1})v = v^2 + fv - v + vf = a^{2n} - a^n.\]

Thus \( g - a^n \in (a^n - a^{2n})R \), and so \( g \in a^n R \) and \( 1 - g \in (1 - a^n)R \).

If there exists an idempotent \( h \in a^n R \) such that \( 1 - h \in (1 - a^n)R \). Write \( h = a^n x, xh = x \). Then \( xa^n = x \). It is easy to verify that \( xa^n = x(a^n x)a^n = a^n x(xa^n) = a^n(xa^n x) = a^n x \). Write \( 1 - h = (1 - a^n)y \), \( y(1 - h) = y \). Likewise, \( y(1 - a^n) = (1 - a^n)y \). One directly checks that \( (a^n - (1 - h))^{-1} = x - y \), i.e., \( a^n - (1 - h) \in U(R) \). By the uniqueness, we get \( 1 - h = f \). Hence, \( g = 1 - f = h \), as desired.

Conversely, assume that (1) and (2) hold. For any \( a \in R \), there exists an \( n \in \mathbb{N} \) and a unique idempotent \( e \in a^n R \) such that \( 1 - e \in (1 - a^n)R \). As in the preceding discussion, we get \( a^n - (1 - e) \in U(R) \). Write \( a^n = f + v \), where \( f = f^2 \), \( v \in U(R) \). Set \( g = 1 - f \). Then \( g = g^2 \in R \). Further, we have \( g \in a^n R \) and \( 1 - g \in (1 - a^n)R \). By the uniqueness, we obtain \( g = e \). Thus, \( f = 1 - e \), hence the result.

**Corollary 2.2.** Let \( R \) be a ring. Then \( R \) is uniquely \( \pi \)-clean if and only if

1. Every idempotent in \( R \) is central.
2. For any \( a \in R \), there exists an \( n \in \mathbb{N} \) and a unique idempotent \( e \in a^n R \) such that \( 1 - e \in (1 - a^n)R \).

**Proof.** Obviously, a ring \( R \) is uniquely \( \pi \)-clean if and only if so is the opposite ring \( R^{op} \). Applying Theorem 2.2 to \( R^{op} \), we complete the proof.

A ring \( R \) is local if it has only one maximal right ideal. A ring \( R \) is potent if for any \( a \in R \) there exists some \( n \in \mathbb{N} \) such that \( a^n = a \). We note that every potent ring is commutative.

**Lemma 2.2.** Let \( R \) be a local ring. If \( R \) is uniquely \( \pi \)-clean, then \( R/J(R) \) is potent.

**Proof.** Suppose that there exists some \( a \in R \) such that \( a^n - a \notin J(R) \) for all \( n \geq 2 \). Then \( a(a^{n-1} - 1) \in U(R) \) as \( R \) is a local ring. This implies that \( a \in U(R) \) and \( a^{n-1} - 1 \in U(R) \) for all \( n \geq 2 \). Since \( R \) is uniquely \( \pi \)-clean, we have an \( m \in \mathbb{N} \) such that \( a^m \in R \) is uniquely clean. But \( a^m = 0 + a^m = 1 + (a^m - 1) \), a contradiction. Therefore, for any \( a \in R \), there exists some integer \( n \geq 2 \) such that \( a^n - a \in J(R) \). That is, \( R/J(R) \) is potent.

\[ \square \]
Let $R$ be a ring. Then $R$ is uniquely $\pi$-clean if and only if

1. $R$ is Abelian;
2. Every idempotent lifts modulo $J(R)$;
3. $R/J(R)$ is potent.\[\frac{1}{2}\]

We have accumulated all the information necessary to prove the following.

**Theorem 2.3.** Let $R$ be a ring. Then the following statements are equivalent:

1. $R$ is uniquely $\pi$-clean.
2. For any $a \in R$, there exist an $m \in \mathbb{N}$ and a central idempotent $e \in R$ such that $a^m - e \in J(R)$.

**Proof.** (1) $\Rightarrow$ (2) In view of Lemma 2.3, $R/J(R)$ is potent. For any $a \in R$, $\bar{a} \in R/J(R)$ is potent, and so $\bar{a}^m \in R/J(R)$ is an idempotent for some $m \in \mathbb{N}$. By using Lemma 2.3 again, we can find a central idempotent $e \in R$ such that $\bar{a}^m = \bar{e}$, and so $a^m - e \in J(R)$.

(2) $\Rightarrow$ (1) If $e \in R$ is an idempotent, then we have a central idempotent $f \in R$ such that $e - f \in J(R)$. As $(e - f)^2 = e - f$, we deduce that $e = f$; hence, every idempotent in $R$ is central. If $e - e^2 \in J(R)$, then we can find a central idempotent $f \in R$ such that $e^m - f \in J(R)$ for some $m \in \mathbb{N}$. As $e - e^2 \in J(R)$, if $m \geq 3$, we see that $e - e^m = (e - e^2) + (e - e^2)e + \cdots + (e - e^2)e^{m-2} \in J(R)$. Thus $e - f \in J(R)$, and then idempotents lift modulo $J(R)$.

For any $a \in R$, there exists $m \in \mathbb{N}$ such that $a^m - e \in J(R)$ for a central idempotent. Hence, $\bar{a}^m = \bar{e}$ in $R/J(R)$. Thus, $S := R/J(R)$ is periodic. Since $S$ is an Abelian exchange ring, if $x^2 = 0$ and $x \neq 0$ in $S$, then $x \notin J(S)$. For any $r \in S$, there exists some idempotent $g \in Srx$ such that $1 - g \in S(1 - rx)$. Write $g = crx$ for $c \in S$. Then $g = g^2 = (crx)g = (cr)gx = (cr)(crx)x = (cr)^2x^2 = 0$, as $S$ is Abelian. Thus, $1 - rx \in S$ is left invertible. Since $S$ is Abelian, it is easy to check that $1 - rx \in U(S)$. This shows that $x \in J(S)$; hence, $x = 0$. This gives a contradiction. Therefore $S$ is reduced.

Let $a \in R$; there exist $m, n$ ($m > n$) such that $\bar{a}^m = \bar{a}^n$ in $S$. Choose $k = n(m - n)$. It is easy to verify that $p = \bar{a}^{k+1}$ is potent and $w = a - \bar{a}^{k+1} \in N(S)$. Further, $\bar{a} = p + w = p$ is potent, and so $S$ is potent. Applying Lemma 2.3, we complete the proof.\[\Box\]

**Corollary 2.3.** Let $R$ be a ring. Then $R$ is uniquely clean if and only if

1. $R$ is uniquely $\pi$-clean;
2. $J(R) = \{x \in R \mid x - 1 \in U(R)\}$.

**Proof.** Obviously, $J(R) \subseteq \{x \in R \mid 1 - x \in U(R)\}$. Suppose that $1 - x \in U(R)$. Then we have an idempotent $e \in R$ and an element $u \in J(R)$ such that $x = e + u$ and $ex = xe$ by [10] Theorem 20]. Thus, $1 - e = (1 - x) + u \in U(R)$, and so $1 - e = 1$. This implies that $e = 0$, whence $x = u \in J(R)$. Therefore $J(R) = \{x \in R \mid 1 - x \in U(R)\}$.

Conversely, assume that (1) and (2) hold. In view of Lemma 2.3, $R/J(R)$ is potent. It follows from $J(R) = \{x \in R \mid x - 1 \in U(R)\}$ that $U(R/J(R)) = \{\bar{1}\}$.\[\Box\]
Write $p = p^n(n \geq 2)$ in $R/J(R)$. Then $(1 - p^{n-1} + p)^{-1} = 1 - p^{n-1} + p^{n-2}$. Hence, $p = p^{n-1}$, and so $p^2 = p^n = p$. This implies that $R/J(R)$ is Boolean. Therefore we complete the proof by Lemma 2.1 and [10] Theorem 20.

**Theorem 2.4.** Let $R$ be a ring. Then $R$ is uniquely \( \pi \)-clean if and only if

1. For any \( a \in R \), there exists an \( m \in \mathbb{N} \) and a unique \( e \in R \) such that \( a^m - e \in J(R) \).
2. \( J(R) = \{ x \in R \mid x^m - 1 \in U(R) \text{ for all } m \in \mathbb{N} \} \).

**Proof.** Suppose that \( R \) is uniquely \( \pi \)-clean. Let \( a \in R \). In view of Theorem [2.3] there exist an \( m \in \mathbb{N} \) and a central idempotent \( g \in R \) such that \( a^m - g \in J(R) \). If there exists an idempotent \( f \in R \) such that \( a^m - f \in J(R) \), then \( g = (a^m - f) - (a^m - g) \in J(R) \). Clearly, \((g - f)^2 = g - f\), and so \( (g - f)(1 - (g - f)^2) = 0 \). Thus, \( g = f \), i.e., the uniqueness is verified.

Clearly, \( J(R) \subseteq \{ x \in R \mid x^m - 1 \in U(R) \text{ for all } m \in \mathbb{N} \} \). If \( x \notin J(R) \), then \( 0 \notin xR \subseteq J(R) \). In view of Lemma [2.1], \( R \) is an exchange ring, and so there exists an idempotent \( 0 \neq e \in xR \). Write \( e = x \) for any \( r \in R \). Choose \( a = exe + 1 - e \).

Then we can find some \( m \in \mathbb{N} \) such that \( a^m \in R \) is uniquely clean. In addition, \( R \) is Abelian by Lemma [2.1]. Obviously, \( a^m = 0 + (exe^m + 1 - e) = e + (exe^m - 1)e + 1 - e \).

If \( x^m - 1 \in U(R) \), then \( 0 = e \), a contradiction. This implies that \( x^m - 1 \notin U(R) \).

That is, \( x \notin \{ x \in R \mid x^m - 1 \in U(R) \text{ for all } m \in \mathbb{N} \} \). Therefore \( \{ x \in R \mid x^m - 1 \in U(R) \} \) for all \( m \in \mathbb{N} \) \( \subseteq J(R) \), as required.

Conversely, assume that (1) and (2) hold. Let \( x \in N(R) \). Then \( x^m - 1 \in U(R) \) for all \( m \in \mathbb{N} \). By hypothesis, we get \( x \in J(R) \). Therefore, every nilpotent element in \( R \) is contained in \( J(R) \). Let \( e \in R \) be an idempotent, and let \( r \in R \). Then \( e + er(1 - e) \in R \) is an idempotent. Hence, there exists a unique \( f \in R \) such that \( (e + er(1 - e)) - f \in J(R) \). By the preceding discussion, \( (e + er(1 - e)) - e = er(1 - e) \in J(R) \). The uniqueness forces \( e = f \). But \( (e+er(1-e)) - (e+er(1-e)) \in J(R) \), and so \( e + er(1 - e) = f = e \). This shows that \( er = ere \). Likewise, \( re = ere \).

That is, \( er = re \), and then \( R \) is Abelian. For any \( a \in R \), there exist an \( m \in \mathbb{N} \) and a unique \( e \in R \) such that \( w := a^m - e \in J(R) \). Then \( a^m = (1 - e) + (2e - 1 + w) \). As \((2e - 1)^2 = 1 \), we see that \( 2e - 1 + w \in U(R) \). If there exists an idempotent \( f \in R \) such that \( a^m - f \in U(R) \), then \( e - f = (a^m - f) - (a^m - e) \in U(R) \). One easily checks that \( (e + f - 1)(e - f)^2 = 0 \), and therefore \( e + f - 1 = 0 \). Thus, \( f = 1 - e \), hence the result.

**Corollary 2.4.** Let \( R \) be a ring. Then \( R \) is uniquely \( \pi \)-clean if and only if

1. For any \( a \in R \), there exist an \( m \in \mathbb{N} \) and a unique \( e \in R \) such that \( a^m - e \in J(R) \).
2. \( N(R) \subseteq J(R) \).

**Proof.** Suppose that \( R \) is uniquely \( \pi \)-clean. (1) is obvious by Theorem 2.4. Let \( a \in N(R) \). Then \( 1 - a^m \in U(R) \) for all \( m \in \mathbb{N} \). It follows by Theorem 2.4 that \( a \in J(R) \). Therefore \( N(R) \subseteq J(R) \).

Conversely, assume that (1) and (2) hold. Let \( e \in R \), and let \( x \in R \). Then \( ex(1 - e) \in J(R) \). By hypothesis, we have some \( m \in \mathbb{N} \) such that the expressions...
\((e + ex(1 - e))^{m} = (e + ex(1 - e)) + 0 = e + ex(1 - e)\) are unique. This implies that \(ex(1 - e)\) = 0, and so \(ex = exe\). Likewise, \(xe = exe\). Therefore \(R\) is Abelian. This yields the result by Theorem 2.3.

**Corollary 2.5.** Let \(R\) be a local ring. Then the following statements are equivalent:

1. \(R\) is uniquely \(\pi\)-clean.
2. \(U(R) = \{x \in R \mid \text{There is an } m \in \mathbb{N} \text{ such that } x^m - 1 \in J(R)\}\).
3. \(J(R) = \{x \in R \mid x^m - 1 \in U(R) \text{ for all } m \in \mathbb{N}\}\).

**Proof.** (1) \(\Rightarrow\) (2) Obviously, \(\{x \in R \mid \text{there is an } m \in \mathbb{N} \text{ such that } x^m - 1 \in J(R)\} \subseteq U(R)\). For any \(x \in U(R), x \notin J(R)\). By hypothesis, there exists some \(m \in \mathbb{N}\) such that \(x^m - 1 \notin U(R)\). As \(R\) is local, \(x^m - 1 \in J(R)\). This implies that \(U(R) \subseteq \{x \in R \mid \text{there is an } m \in \mathbb{N} \text{ such that } x^m - 1 \notin J(R)\}\), as required.

(2) \(\Rightarrow\) (1) For any \(x \in R\), we see that either \(x \in J(R)\) or \(x \in U(R)\). This implies that \(\bar{x} = 0\) or \(x^m = 1\) in \(R/J(R)\). This completes the proof.

\(\square\)

### 3. Factors of Prime Ideals

The aim of this section is to characterize uniquely \(\pi\)-clean rings by means of prime ideals containing the Jacobson radicals. We use \(J\)-spec\((R)\) to denote the set \(\{P \in \text{Spec}(R) \mid J(R) \subseteq P\}\). Obviously, every maximal ideal is contained in \(J\)-spec\((R)\). Set

\[J^{\ast}(R) = \bigcap\{P \mid P \text{ is a maximal ideal of } R\}.\]

We will see that \(J(R) \subseteq J^{\ast}(R)\). In general, they are not the same. For instance, \(J(R) = 0\) and \(J^{\ast}(R) = \{x \in R \mid \dim_{F}(xV) < \infty\}\), where \(R = \text{End}_{F}(V)\) and \(V\) is an infinite-dimensional vector space over a field \(F\). Furthermore, we characterize a uniquely \(\pi\)-clean ring \(R\) by means of the radical-like ideal \(J^{\ast}(R)\). A ring \(R\) is strongly \(\pi\)-regular if, for any \(a \in R\) there exists \(n \in \mathbb{N}\) such that \(a^n \in a^{n+1}R\). We have

**Lemma 3.1.** [7, Corollary 2.8] Let \(R\) be a commutative ring. Then the following statements are equivalent:

1. \(R\) is strongly \(\pi\)-regular.
2. \(R\) is an exchange ring in which every prime ideal of \(R\) is maximal.

**Lemma 3.2.** Let \(R\) be an Abelian exchange ring, and let \(x \in R\). Then \(RxR = R\) if and only if \(x \in U(R)\).

**Proof.** If \(x \in U(R)\), then \(RxR = R\). Conversely, assume that \(RxR = R\). As in the proof of Proposition 17.1.9], there exists an idempotent \(e \in R\) such that \(e \in xR\) such that \(R \subseteq R\). This implies that \(e = 1\). Write \(xy = 1\). Then \(yx = y(xy)x = (yx)^2\). Hence, \(yx = y(yx)x\). Therefore \(1 = x(yx)y = xy(yx)xy = yx\), and so \(x \in U(R)\). This completes the proof. \(\square\)
Herstein’s theorem says that a ring $R$ is periodic if and only if for any $a \in R$, there exists $n \in \mathbb{N}$ such that $a^n = a^{n+1} f(a)$ for some $f(t) \in \mathbb{Z}[t]$. We recall that a ring $R$ is torsion, provided that for any nonzero $a \in R$ there exists $m \in \mathbb{N}$ such that $a^m = 1$. With this information we now derive

**Theorem 3.1.** Let $R$ be a ring. Then $R$ is uniquely $\pi$-clean if and only if

1. $R$ is Abelian;
2. Every idempotent lifts modulo $J(R)$;
3. $R/P$ is torsion for all $P \in J$-spec($R$).

**Proof.** Suppose $R$ is uniquely $\pi$-clean. In view of Lemmas 2.1 and 2.3, $R$ is an Abelian exchange ring, and $R/J(R)$ is potent. Let $P \in J$-spec($R$). Then $R/J(R)/P/J(R) \cong R/P$ is prime: hence, $P/J(R)$ is a prime ideal of $R/J(R)$. As every potent ring is commutative, $R/J(R)$ is a commutative $\pi$-regular ring. It follows from Lemma 3.1 that $P/J(R)$ is a maximal ideal of $R/J(R)$. We infer that $P$ is a maximal ideal of $R$.

Clearly, $R := R/P$ is an Abelian exchange ring. Since $P$ is maximal, $R/P$ is simple. For any $0 \neq x \in R$, we have $RxR = R$. By Lemma 3.2, $x \in U(R/P)$. Hence, $R/P$ is a division ring. On the other hand, $R/P \cong R/J(R)/P/J(R)$ is potent. Thus, we have some $m \in \mathbb{N}$ such that $x^{m+1} = x$, and so $x^m = 1$. This implies that $R/P$ is torsion, as required.

Conversely, assume that (1)–(3) hold. Assume that $R$ is not uniquely $\pi$-clean. Set $S = R/J(R)$. In view of Theorem 2.3, $S$ is not periodic. By using Herstein’s theorem, there exists some $a \in S$ such that $a^m \neq a^{m+1} f(a)$ for any $m \in \mathbb{N}$ and any $f(x) \in \mathbb{Z}[x]$. Let $\Omega = \{I \subseteq S \mid \bar{a}^m \neq \bar{a}^{m+1} f(\bar{a})\}$ in $S/I$ for any $m \in \mathbb{N}$ and any $f(x) \in \mathbb{Z}[x]$. Then $\Omega$ is an nonempty inductive. By using Zorn’s lemma, there exists an ideal $Q$ of $S$ which is maximal in $\Omega$. If $Q$ is not prime, then there exist two ideals $K$ and $L$ of $R$ such that $K, L \not\subseteq Q$ and $KL \subseteq Q$. By the maximality of $Q$, we can find some $s, t \in \mathbb{N}$ and some $f(x), g(x) \in \mathbb{Z}[x]$ such that $\bar{a}^s = \bar{a}^{s+1} f(\bar{a})$ in $R/(K + Q)$ and $\bar{a}t = \bar{a}^{t+1} g(\bar{a})$ in $R/(L + Q)$. Thus, $a^s - a^{s+1} f(a) \in K + Q$ and $a^t - a^{t+1} g(a) \in L + Q$, and so $(a^s - a^{s+1} f(a))(a^t - a^{t+1} g(a)) \in (K + Q)(L + Q) \subseteq KL + Q \subseteq Q$. In $S/Q$, we have $\bar{a}^{s+t} = \bar{a}^{s+t+1} h(\bar{a})$ for some $h(x) \in \mathbb{Z}[x]$. This contradicts the choice of $Q$. Hence, $Q \not\subseteq J$-spec($R$). By hypothesis, $R/Q$ is torsion, and so $R/Q$ is periodic, which is impossible. Therefore $R$ is uniquely $\pi$-clean. □

**Corollary 3.1.** A ring $R$ is uniquely clean if and only if

1. $R$ is uniquely $\pi$-clean.
2. $R/M \cong \mathbb{Z}_2$ for all maximal ideals $M$ of $R$.

**Proof.** Suppose $R$ is uniquely clean. Then $R$ is uniquely $\pi$-clean. (2) is proved by [3] Theorem 2.1.

Conversely, assume that (1) and (2) hold. For all maximal ideals $M$ of $R$, $1_{R/M}$ is not the sum of two units in $R/M$. In view of Lemma 2.1, $R$ is an Abelian exchange ring, and so it is clean. Let $x \in R$. Write $x = e_1 + u_1 = e_2 + u_2$, $e_1 = e_1^2$, $e_2 = e_2^2$ and $u_1, u_2 \in U(R)$. If $R(1 - e_2(1 - e_1))R \neq R$, then there exists a maximal ideal $M$ of $R$ such that $R(1 - e_2(1 - e_1))R \subseteq M$. Clearly,
$J(R) \subseteq M$. Hence, $\bar{x} = \bar{x}_1 + \bar{x}_2 = \bar{e}_2 + \bar{u}_2$ in $R/M$. By Theorem 3.1, $R/M$ is a division ring. This implies that $\bar{e}_1$ are 0 or 1. If $\bar{e}_1 \neq \bar{e}_2$, then $1_{R/M}$ is the sum of two units, a contradiction. Therefore we get $e_1 = e_2 \in M$. This implies that $e_2(1-e_1) = (e_1-e_2)(e_1-1) \in M$, and so $1 = e_2(1-e_1) + (1-e_2(1-e_1)) \in M$, a contradiction. As a result, $R(1-e_2(1-e_1))R = R$. As $e_2(1-e_1) \in R$ is an idempotent, we get $e_2(1-e_1) = 0$, and so $e_2 = e_2e_1$. Likewise, $e_1 = e_1e_2$. Consequently, $e_1 = e_2$, and then $u_1 = u_2$. Therefore $R$ is uniquely clean.

Let $S(R)$ be the nonempty set of all ideals of a ring $R$ generated by central idempotents. By Zorn’s lemma, $S(R)$ contains maximal elements. As usually, we say that $R/P$ is a Pierce stalk if $P$ is a maximal element of the set $S(R)$, and that $P$ is a Pierce ideal. Let $\text{Pier}(R)$ be the set of all Pierce ideals of $R$.

**Proposition 3.1.** Every uniquely $\pi$-clean ring is the subdirect product of rings $R_i$, where each $R_i/\text{P}(R_i)$ is torsion.

**Proof.** Let $R$ be a uniquely $\pi$-clean ring. [9] Remark 11.2] says that the intersection of all Pierce ideals of $R$ is zero, i.e., $\bigcap \{ P \mid P \in \text{Pier}(R) \} = 0$. Let $\varphi_P : R \to R/P$ be the natural epimorphism. Then $\bigcap_{P \in \text{Pier}(R)} \ker \varphi_P = \bigcap_{P \in \text{Pier}(R)} P = 0$. Hence, $R$ is the subdirect product of all $R/P$, where $P \in \text{Pier}(R)$. In view of Lemma 2.1, $R$ is an Abelian exchange ring. Let $P \in \text{Pier}(R)$. Then $R/P$ is an exchange ring. As $R$ is indecomposable, we see that $R/P$ is a local ring. By an argument in [9], $R/P$ is uniquely $\pi$-clean, and so $R/P/\text{P}(R/P)$ is potent from Lemma 2.3 as needed. 

**Lemma 3.3.** Let $R$ be an Abelian exchange ring. Then $J^*(R) = J(R)$.

**Proof.** Let $M$ be a maximal ideal of $R$. If $J(R) \not\subseteq M$, then $J(R) + M = R$. Write $x + y = 1$ with $x \in J(R)$, $y \in M$. Then $y = 1 - x \in U(R)$, an absurd. Hence, $J(R) \subseteq M$. This implies that $J(R) \subseteq J^*(R)$. Let $x \in J^*(R)$, and let $r \in R$. If $R(1 - xr)R \neq R$, then we can find a maximal ideal $M$ of $R$ such that $R(1 - xr)R \subseteq M$, and so $1 - xr \in M$. It follows that $1 = xr + (1 - xr) \in M$, which is impossible. Therefore $R(1 - xr)R = R$. In light of Lemma 3.2, $1 - xr \in U(R)$, and then $x \in J(R)$. This completes the proof.

**Theorem 3.2.** Let $R$ be a ring. Then $R$ is uniquely $\pi$-clean if and only if

1. $R$ is an exchange ring;
2. $R/J^*(R)$ is potent and every idempotent uniquely lifts modulo $J^*(R)$.

**Proof.** Suppose $R$ is uniquely $\pi$-clean. Then $R$ is an Abelian exchange ring by Lemma 2.1. In view of Lemma 3.3, $J^*(R) = J(R)$. It follows from Lemma 2.3 that $R/J^*(R)$ is potent. Let $e - e^2 \in J(R)$. Then we can find an idempotent $f \in R$ such that $e - f \in J(R)$. Since $(e - f)^2(1 - (e - f)) = 0$, we get $e = f$, as desired.

Conversely, assume that (1) and (2) hold. Let $e \in R$ be an idempotent, and let $r \in R$. Then $e(r(1 - e)) \in R/J^*(R)$ is potent. This implies that $e(r(1 - e)) = 0$, and so $er(1 - e) \in J^*(R)$. Since $e - e \in J^*(R)$, by the uniqueness, we get $e = e + er(1 - e)$, and so $er = ere$. Likewise, $re = ere$; hence that $er = re$. 


Thus, $R$ is Abelian. In light of Lemma 3.3, $J^*(R) = J(R)$. Therefore we complete the proof, in terms of Lemma 2.3.

**Corollary 3.2.** Let $R$ be a ring which has finitely many maximal ideals. Then $R$ is uniquely $\pi$-clean if and only if

1. $R$ is an exchange ring;
2. $R/J^*(R)$ is the direct sum of finitely many torsion rings and every idempotent uniquely lifts modulo $J^*(R)$.

**Proof.** $\Rightarrow$: Clearly, $R$ is an exchange ring. Let $M$ be a maximal ideal of $R$. As in the proof of Lemma 3.3, we see that $J(R) \subseteq M$. This shows that $M \in J$-$\text{spec}(R)$. Therefore $R/M$ is torsion by Theorem 3.1. Since $R$ has finitely many maximal ideals $M_1, \ldots, M_n$, we see that $R/J^*(R) \cong R/M_1 \oplus \cdots \oplus R/M_n$. Therefore $R/J^*(R)$ is the direct sum of finitely many torsion rings, as desired.

$\Leftarrow$: As every torsion ring is potent, we see that $R/J^*(R)$ is potent. Therefore we complete the proof, by Theorem 3.2.

**Theorem 3.3.** Let $R$ be a ring. Then $R$ is uniquely $\pi$-clean if and only if

1. For any $a \in R$, there exist an $m \in \mathbb{N}$ and a unique $e \in R$ such that $a^m - e \in J^*(R)$.
2. $J^*(R) = \{x \in R \mid x^m - 1 \in U(R) \text{ for all } m \in \mathbb{N}\}$.

**Proof.** One direction is obvious by Lemma 3.3 and Theorem 2.4. Conversely, assume that (1) and (2) hold. Let $x \in N(R)$. Then $x^m - 1 \in U(R)$ for all $m \in \mathbb{N}$. By hypothesis, we have $x \in J^*(R)$, and so $N(R) \subseteq J^*(R)$. Let $e \in R$ be an idempotent, and let $r \in R$. Then $e + er(1 - e) + 0 = e + er(1 - e)$ with $0, er(1 - e) \in J^*(R)$. By the uniqueness, we get $er = ecr$. Similarly, we have $re = cere$. That is, $er = re$. We infer that $R$ is Abelian. For any $a \in R$, there exist an $m \in \mathbb{N}$ and a unique $e \in R$ such that $w := a^m - e \in J^*(R)$. Then $a^m = (1 - e) + (2e - 1 + w)$. But $2e - 1 + w = (1 - 2e)(1 - 2e)w - 1) \in U(R)$, by (2). If there exists an idempotent $f \in R$ such that $a^m - f \in U(R)$, then $e - f = (a^m - f) - (a^m - e) = (a^m - f)(1 - (a^m - f)^{-1}(a^m - e)) \in U(R)$. It follows from $(e + f - 1)(e - f)^2 = 0$ that $f = 1 - e$, and we are through.

Let $P(R)$ be the intersection of all prime ideals of $R$, i.e., $P(R)$ is the prime radical of $R$. As is well known, $P(R)$ is the intersection of all minimal prime ideals of $R$.

**Corollary 3.3.** Let $R$ be a uniquely $\pi$-clean in which every prime ideal is maximal. Then $P(R) = \{x \in R \mid x^m - 1 \in U(R) \text{ for all } m \in \mathbb{N}\}$.

**Proof.** As every maximal ideal is prime, we deduce that $J^*(R) = P(R)$, and therefore we complete the proof by Theorem 3.3.
4. Certain Classes

In this section we investigate certain classes of uniquely $\pi$-clean rings. We now recall the concept of ideal-extensions. Let $R$ be a ring with an identity and $S$ be a ring (not necessary unitary), and let $S$ be an $R$-$R$-bimodule in which $(s_1s_2)r = s_1(s_2r)$, $r(s_1s_2) = (rs_1)s_2$ and $(s_1r)s_2 = s_1(rs_2)$ for all $s_1, s_2 \in S$, $r \in R$. The ideal extension $I(R; S)$ is defined to be the additive Abelian group $R \oplus S$ with multiplication $(r_1, s_1)(r_2, s_2) = (r_1r_2, s_1s_2 + r_1s_2 + s_1r_2)$ (see [10]). We start this section by examining when an ideal extension is uniquely $\pi$-clean.

**Theorem 4.1.** The ideal-extension $I(R; S)$ is uniquely $\pi$-clean and $S$ is idempotent-free if and only if

1. $R$ is uniquely $\pi$-clean;
2. If $e = e^2 \in R$, then $es = se$ for all $s \in S$;
3. If $s \in S$, then there exists an $s' \in S$ such that $ss' = s's$ and $s + s' + ss' = 0$.

**Proof.** Assume that (1)–(3) hold. Let $e \in S$ be an idempotent. Then $(-e) + s' + (-e)s' = 0$ for some $s' \in S$. Hence, $(1 - e)(1 + s') = 1$, and so $e = 0$. That is, $S$ is idempotent-free. Let $(a, s) \in I(R; S)$. Then $a \in R$ is uniquely $\pi$-clean. Thus, we have some $n \in \mathbb{N}$ such that $a^n \in R$ is uniquely clean. Write $a^n = e + u$, $e = e^2 \in R$, $u \in U(R)$. Hence, $(a, s)^n = (a^n, x) = (e, 0) + (u, x)$ for some $x \in S$. Clearly, $(e, 0)^2 = (e, 0)$. As $x \in S$, we see that $u^{-1}x \in S$, and so we have some $t \in S$ such that $u^{-1}x + t + u^{-1}xt = 0$ and $u^{-1}xt = tu^{-1}x$. This implies that $1 + u^{-1}x = (1 + t)^{-1} \in U(R)$. One easily checks that $(u, x)^{-1} = (u^{-1}, -(1 + u^{-1}x)^{-1}u^{-1}ux^{-1})$; hence, $(u, x) \in U(I(R; S))$. Write $(a, s)^n = (f, y) + (v, w), (f, y)^2 = (f, y)$ and $(v, w) \in U(I(R; S))$. Then $f = f^2 \in R, y = 0$ and $v \in U(R)$. Clearly, $a^n = f + v$. Further, $x = y + w = w$. This implies that $f = e, v = u$, and so $(f, y) = (e, 0), (v, w) = (u, x)$. As a result, $(a, s) \in I(R; S)$ is uniquely $\pi$-clean, and so $I(R; S)$ is uniquely $\pi$-clean.

Assume that $I(R; S)$ is uniquely $\pi$-clean and $S$ is idempotent-free. Then $R$ is uniquely $\pi$-clean. Let $e = e^2 \in R$ and $s \in S$. In view of Lemma 2.1, $(e, 0)^2 \in I(R; S)$ is central. Hence, $(e, 0)(0, s) = (0, s)(e, 0)$, and so $es = se$. For any $s \in S$, there exists some $n \in \mathbb{N}$ such that $(1, s)^n \in I(R; S)$ is uniquely clean. Write $(1, s)^n = (1, x) = (f, y) + (u, v)$ where $x \in S$, $(f, y) \in I(R; S)$ is an idempotent and $(u, v) \in I(R; S)$ is a unit. Clearly, $f = 0$, and so $y = 0$. This implies that $x = y + v = v$; hence, $(1, x) \in I(R; S)$ is a unit. Further, $(1, s) \in I(R; S)$ is a unit. Write $(1, s)^{-1} = (1, s')$ for a $s' \in S$. Then $ss' = s's$ and $s + s' + ss' = 0$, and hence the result.

**Corollary 4.1.** Let $R$ be uniquely $\pi$-clean. Then $S = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$ is uniquely $\pi$-clean.

**Proof.** Let $T = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn} = 0\}$. Then $S \cong I(R; T)$. Then the result follows by Theorem 4.1.

A ring $R$ is called potently $J$-clean if for any $a \in R$ there exists a potent $p \in R$ such that $a - p \in J(R)$. We shall show that such rings form a subclass of uniquely
π-clean rings. A ring $R$ is an exchange ring if and only if $R/J(R)$ is an exchange ring, and every idempotent lifts modulo $J(R)$. We have

**Lemma 4.1.** Every potently $J$-clean ring is an exchange ring.

**Proof.** Let $R$ be a potently $J$-clean ring. Then $R/J(R)$ is potent, and so it is an exchange ring. Let $π ∈ R/J(R)$ be an idempotent. Then we have a potent $p ∈ R$ such that $w := c − p ∈ J(R)$. Write $p = p^n$ for some $n ≥ 2$. Then $p^{n−1} ∈ R$ is an idempotent. Moreover, $e = p + w$, and so $e^{n−1} = p^{n−1} + v$ for some $v ∈ J(R)$. But $e − e^{n−1} ∈ J(R)$. Hence, $e = p^{n−1} = (e − e^{n−1}) + (e^{n−1} − p^{n−1}) ∈ J(R)$. So idempotents can be lifted modulo $J(R)$. Therefore $R$ is an exchange ring. □

**Theorem 4.2.** Every Abelian potently $J$-clean ring is uniquely $π$-clean.

**Proof.** Let $R$ be an Abelian potently $J$-clean ring. Then $R$ is an exchange ring by Lemma 4.1. Thus, every idempotent in $R$ lifts modulo $J(R)$. For any $a ∈ R$, there exists a potent $p ∈ R$ such that $a − p ∈ J(R)$. This implies that $a ∈ R/J(R)$ is potent, and so $R/J(R)$ is potent. According to Lemma 2.3, $R$ is uniquely $π$-clean.

**Corollary 4.2.** Let $R$ be Abelian. If for any sequence of elements $\{a_i\} ⊆ R$ there exists a $k ∈ \mathbb{N}$ and $n_1, \ldots, n_k ≥ 2$ such that $(a_1 - a_1^{n_1}) \cdots (a_k - a_k^{n_k}) = 0$, then $R$ is uniquely $π$-clean.

**Proof.** For any $a ∈ R$, we have a $k ∈ \mathbb{N}$ and $n_1, \ldots, n_k ≥ 2$ such that $(a - a^{n_1}) \cdots (a - a^{n_k}) = 0$. This implies that $a^k = a^{k+1}f(a)$ for some $f(t) ∈ \mathbb{Z}[t]$. By Herstein’s theorem, $R$ is periodic. Therefore every element in $R$ is the sum of a potent element and a nilpotent element.

Clearly, $R/J(R)$ is isomorphic to a subdirect product of some primitive rings $R_i$. Case 1. There exists a subring $S_i$ of $R_i$ which admits an epimorphism $ϕ_i : S_i → M_2(D_i)$ where $D_i$ is a division ring.

Case 2. $R_i ≅ M_{m_i}(D_i)$ for a division ring $D_i$. Clearly, the hypothesis is inherited by all subrings, all homomorphic images and all corners of $R$, we claim that, for any sequence of elements $\{a_s\} ⊆ M_2(D_i)$ there exists $s ∈ \mathbb{N}$ and $m_1, \ldots, m_s ≥ 2$ such that $(a_1 - a_1^{m_1}) \cdots (a_s - a_s^{m_s}) = 0$. Choose $a_i = e_{12}$ if $i$ is odd and $a_i = e_{21}$ if $i$ is even. Then $(a_1 - a_1^{m_1})(a_2 - a_2^{m_2}) \cdots (a_s - a_s^{m_s}) = a_1a_2 \cdots a_s ≠ 0$, a contradiction. This forces $m_i = 1$ for all $i$. We infer that each $R_i$ is reduced, and then so is $R/J(R)$. If $a ∈ N(R)$, we have some $n ∈ \mathbb{N}$ such that $a^n = 0$, and thus $a^n = 0$ is $R/J(R)$.

Hence, $a ∈ J(R/J(R)) = 0$. This implies that $a ∈ J(R)$, and so $N(R) ⊆ J(R)$. Therefore $R$ is potently $J$-clean, hence the result by Theorem 4.2. □

## 5. Uniquely $π$-nil Clean Rings

In this section, we explore uniquely $π$-nil-clean rings, and completely determine when a ring is uniquely $π$-nil-clean ring.

**Lemma 5.1.** Let $R$ be a ring. Then the following statements are equivalent:

1. $R$ is uniquely $π$-nil-clean.
2. $R$ is an Abelian periodic ring.
Proof. (1) $\Rightarrow$ (2). Let $e \in R$ be an idempotent and $r \in R$. Choose $a = e + er(1-e)$. Then we can find some $m \in \mathbb{N}$ such that $a^m \in R$ is uniquely nil clean. As $a = a^m = e + er(1-e) = (e + er(1-e)) + 0$, by the uniqueness, we get $er(1-e) = 0$, and so $er = ere$. Likewise, $re = ere$, and so $er = re$. Therefore $R$ is Abelian. Let $a \in R$. Then there exists some $n \in \mathbb{N}$ such that $a^n = f + u$, where $f = f^2 \in R$ and $u \in N(R)$. Hence, $a^{2n} = f + v$ for a $v \in N(R)$ and $w = vu$. This shows that $a^n - a^{2n} = u - v \in N(R)$. Thus, we have a $k \in \mathbb{N}$ such that $a^{nk} = a^{nk+1}f(a)$ for some $f(t) \in \mathbb{Z}[t]$. In light of Herstein’s theorem, $R$ is periodic.

(2) $\Rightarrow$ (1) Let $a \in R$. Since $R$ is periodic, there exists some $m \in \mathbb{N}$ such that $a^m \in R$ is an idempotent. Write $a^m = e + w$ where $e = e^2 \in R$ and $w \in N(R)$. Then $a^m - e = w \in N(R)$. As $R$ is Abelian, we see that $(a^m - e)^2 = a^m - e$. Thus, $(a^m - e)(1 - (a^m - e)^2) = 1$, and so $a^m = e$, as required.

As every finite ring is periodic, it follows from Lemma 5.1 that every finite commutative ring is uniquely $\pi$-nil-clean, e.g., $\mathbb{Z}_n[a] = \{a + bx \mid a, b \in \mathbb{Z}_n, \alpha = -\frac{1}{2} + \frac{\sqrt{-3}}{2}, \ i^2 = -1\}$.

The above observation leads us to the following result alluded to earlier.

Theorem 5.1. Let $R$ be a ring. Then the following are equivalent:

1. $R$ is uniquely $\pi$-nil-clean.
2. $R$ is uniquely $\pi$-clean and $J(R)$ is nil.
3. For any $a \in R$, there exist some $m \in \mathbb{N}$ and a unique idempotent $e \in R$ such that $a^m - e \in P(R)$.

Proof. (1) $\Rightarrow$ (3). By virtue of Lemma 5.1, $R$ is an Abelian periodic ring. In view of [2] Theorem 2, $N(R)$ forms an ideal of $R$, and so $N(R) = P(R)$. For any $a \in R$, there exists some $m \in \mathbb{N}$ such that $a^m$ is uniquely nil clean. Write $a^m = e + w$ with $e = e^2$ and $w \in N(R)$. Therefore $a^m - e \in P(R)$, as required.

(3) $\Rightarrow$ (2). Let $e \in R$ be an idempotent, and let $r \in R$. Then we have an idempotent $f \in R$ such that $er(1-e) = f + w$ for a $w \in P(R)$. Hence, $1 - f = 1 - er(1-e) + w = (1 - er(1-e))(1 + (1 + er(1-e))w) \in U(R)$. We infer that $f = 0$, and so $er(1-e) = w \in P(R)$. But we have a unique expression $e + er(1-e) = e + er(1-e) + 0$ where $er(1-e), 0 \in P(R)$. By the uniqueness, we get $e = e + er(1-e)$, and so $er = re$. Similarly, $re = er$, i.e., $R$ is Abelian.

Let $x \in J(R)$. Write $x = h + v$ with $h = h^2 \in R, v \in P(R)$. Then $h = x - v \in J(R)$; hence that $h = 0$. It follows that $J(R) = P(R)$. Accordingly, for any $a \in R$, there exist some $m \in \mathbb{N}$ and a unique idempotent $e \in R$ such that $a^m - e \in J(R)$.

If $x \in N(R)$, then we have an idempotent $g \in R$ and a $u \in P(R)$ such that $x = g + u$, and so $g = x - u$. As $R$ is Abelian, we see that $xu = ux$, and then $g \in N(R)$. This shows that $g = 0$. Consequently, $x = u \in P(R) \subseteq J(R)$. We infer that $N(R) \subseteq J(R)$. In light of Corollary 2.4, $R$ is uniquely $\pi$-clean, as desired.

(2) $\Rightarrow$ (1). In view of Lemma 2.1, $R$ is Abelian. In view of Lemma 2.3, $R/J(R)$ is potent. Let $a \in R$. Then $\overline{a} = a^m(m \geq 2)$, and so $a - a^m \in J(R)$. As $J(R)$ is nil, every idempotent lifts modulo $J(R)$. Hence, we can find some $n \in \mathbb{N}$ such that
(a − am)n = 0, and so an = a(n+1)f(t) for some f(t) ∈ Z[t]. In terms of Herstein’s theorem, R is periodic. This completes the proof, by Lemma 5.1

**Corollary 5.1.** Let R be a ring. Then the following statements are equivalent:
1. R is uniquely π-nil-clean.
2. R/J(R) is potent, R is Abelian and J(R) is nil.
3. For any a ∈ R, there exists some m ∈ N and a central idempotent e ∈ R such that am − e ∈ P(R).

**Proof.** (1) ⇒ (2) is proved by Theorem 5.1 and Lemma 2.3
(1) ⇒ (3) This is obvious, in view of Lemma 5.1 and Theorem 5.1
(3) ⇒ (1). For any a ∈ R, there exist some m ∈ N and a central idempotent e ∈ R such that am − e ∈ P(R). Write am − f ∈ P(R) for an idempotent f ∈ R. Then e − f = (am − f) − (am − e) ∈ P(R). As (e − f)3 = e − f, we conclude that e = f, and we are through by Theorem 5.1.

Let n > 2 be a fixed integer. Following Yaqub [8], a ring R is said to be generalized n-like provided that for any a, b ∈ R, (ab)n − abn − anb + ab = 0.

**Proposition 5.1.** Every generalized n-like ring is uniquely π-nil-clean.

**Proof.** Let a ∈ R. Then a2n − 2an+1 + a2 = 0, and so (a − a)n2 = 0. Thus, a − an ∈ N(R). Hence, an = a(n+1)f(a) for some f(t) ∈ Z[t]. Accordingly, R is periodic by Herstein’s theorem.

Let e, f ∈ R. Since R is a generalized n-like ring, we have

\[(1 − e)f \]n = (1 − e)fe + (1 − e)fe = 0;

\[(1 − e)f \]n = (1 − e)f + (1 − e)f − (1 − e)f = (1 − e)f.

Reiterating in the last, we get (1 − e)f = ((1 − e)f)2n, and so (1 − e)f = 0. Hence, ef = efe. Likewise, ef = efe. Therefore ef = efe. We infer that R is Abelian.

Therefore we conclude that R is uniquely π-nil-clean, in terms of Lemma 5.1.

Let R = \( \left\{ \begin{pmatrix} z & y & z \\ x & z & y \\ 0 & 0 & y \end{pmatrix} \right\} | x, y, z \in GF(4) \). It is easy to check that for each a ∈ R, a2 = a or a2 = a2 = 0. Therefore R is a generalized 7-like ring. By Proposition 5.1, R is uniquely π-clean which is a noncommutative periodic ring.

An element a ∈ R is uniquely weakly nil-clean provided that a or −a is uniquely nil-clean. A ring R is uniquely weakly nil-clean ring provided that every element in R is uniquely weakly nil-clean [5].

**Lemma 5.2.** Every uniquely weakly nil-clean ring is uniquely π-nil-clean.

**Proof.** Let R be a uniquely weakly nil-clean ring. In view of [5] Theorem 12, R is Abelian. Let a ∈ R. Then there exists an idempotent e ∈ R such that a − e ∈ N(R) or −a − e ∈ N(R). If a − e ∈ N(R), then a − a2 ∈ N(R). If −a − e ∈ N(R), then a + a2 ∈ N(R). In any case, an = a(n+1)f(t) for some f(t) ∈ Z[t]. In view of Herstein’s theorem, R is periodic. Therefore R is uniquely π-nil-clean, in terms of Lemma 5.1.
Theorem 1.12] says that a ring $R$ is a uniquely weakly nil-clean ring if and only if $R$ is Abelian, $J(R)$ is nil and $R/J(R)$ is a Boolean ring, $\mathbb{Z}_3$ or the product of two such rings. We have

**Theorem 5.2.** A ring $R$ is a uniquely weakly nil-clean ring if and only if

1. $R$ is uniquely $\pi$-nil-clean;
2. $R/J(R)$ is a Boolean ring, $\mathbb{Z}_3$ or the product of two such rings.

**Proof.** $\Rightarrow$: In view of Lemma 5.2 proving (1). Further, proving (2) in terms of [4] Theorem 18.

$\Leftarrow$: As $R$ is uniquely $\pi$-nil-clean, in view of Corollary 5.1, $R$ is Abelian and $J(R)$ is nil, then by (2) and in light of [5] Theorem 18, $R$ is a uniquely weakly nil-clean ring.

**Corollary 5.2.** A ring $R$ is a uniquely weakly nil-clean ring if and only if for any $a \in R$, there exists a central idempotent $e \in R$ such that $a\pm e \in P(R)$ or $a\pm e \in P(R)$.

**Proof.** $\Rightarrow$: In view of Corollary 5.1 $R$ is uniquely $\pi$-nil-clean. For any $a \in R$, by hypothesis, we see that $a$ or $-a$ is an idempotent in $R/J(R)$. By virtue of [1] Theorem 1.12, $R/J(R)$ is a Boolean ring, $\mathbb{Z}_3$ or the product of two such rings.

$\Leftarrow$: Let $a \in R$. By (2), there exists a central idempotent $e \in R$ such that $a\pm e \in P(R)$ or $a\pm e \in P(R)$. Hence, $a^2\pm e = (a\pm e)(a\pm e) \in P(R)$. Thus, $R$ is uniquely $\pi$-nil-clean, by Theorem 5.1. Let $x \in J(R)$. Then there exists a central idempotent $f \in R$ such that $x\mp f$ or $x\mp f$ is in $P(R)$. If $x\mp f \in P(R)$, then $f \in J(R)$, and so $f = 0$. This implies that $x \in P(R)$. If $x\mp f \in J(R)$, similarly, $x \in P(R)$. Hence, $J(R) \subseteq P(R)$. We infer that $J(R) = P(R)$. Thus, $R/J(R)$ is a Boolean ring, $\mathbb{Z}_3$ or the product of two such rings, by [1] Theorem 1.12. In light of Theorem 5.2, the result follows.

A ring $R$ is uniquely nil-clean provided that every element in $R$ is uniquely nil-clean. [5] Corollary 13] says that $R$ is a uniquely nil-clean ring if and only if $R$ is a uniquely weakly nil-clean ring and $2 \in J(R)$. Further, we derive

**Corollary 5.3.** A ring $R$ is a uniquely nil-clean ring if and only if

1. $R$ is uniquely $\pi$-nil-clean;
2. $R/J(R)$ is a Boolean ring.

**Proof.** $\Rightarrow$: Clearly, $R$ is uniquely $\pi$-nil-clean. In view of [3] Theorem 4.5, $R/J(R)$ is Boolean.

$\Leftarrow$: By virtue of Theorem 5.2, $R$ is a uniquely weakly nil-clean ring. As $2^2 = 2$, we see that $2 \in J(R)$. Therefore $R$ is a uniquely nil-clean ring, in terms of [5] Corollary 13.

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References


Department of Mathematics (Received 31 05 2015) Hangzhou Normal University (Revised 09 01 2016)
Hangzhou China
huanyinchen@aliyun.com

Women’s University of Semnan (Farzanegan) Iran

Women’s University of Semnan (Farzanegan) Iran

m.sheibani@gmail.com