A NEW THEOREM ON ABSOLUTE MATRIX SUMMABILITY OF FOURIER SERIES

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Abstract. We generalize a main theorem dealing with absolute weighted mean summability of Fourier series to the $|A, p_n|_k$ summability factors of Fourier series under weaker conditions. Also some new and known results are obtained.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums $(s_n)$. By $u_n^\alpha$ and $t_n^\alpha$ we denote the $n$th Cesàro means of order $\alpha$, with $\alpha > -1$, of the sequence $(s_n)$ and $(na_n)$, respectively, that is (see [6])

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \quad \text{and} \quad t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v,$$

where

$$A_n^\alpha = \frac{(\alpha + 1)(\alpha + 2) \ldots (\alpha + n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0.$$

The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$, if (see [8,10])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} \frac{1}{n} |u_n^\alpha|^k < \infty.$$

If we take $\alpha = 1$, then $|C, \alpha|_k$ summability reduces to $|C, 1|_k$ summability.

Let $(p_n)$ be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \ i \geq 1).$$

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The sequence-to-sequence transformation $t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$ defines the sequence $(t_n)$ of the Riesz mean or simply the $(\tilde{N}, p_n)$ mean of the sequence $(s_n)$ generated by the sequence of coefficients $(p_n)$ (see [9]).

The series $\sum a_n$ is said to be summable $|\tilde{N}, p_n|_k$, $k \geq 1$, if (see [1])
$$\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{k-1} |t_n - t_{n-1}|^k < \infty.$$  

In the special case when $p_n = 1$ for all values of $n$ (resp. $k = 1$), $|\tilde{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (resp. $|\tilde{N}, p_n|$) summability.

2. Known Results

The following theorems are dealing with $|\tilde{N}, p_n|_k$ summability factors of infinite series.

**Theorem 2.1.** [2] Let $(p_n)$ be a sequence of positive numbers such that
$$P_n = O(np_n) \quad \text{as} \quad n \to \infty.$$  
Let $(X_n)$ be a positive monotonic nondecreasing sequence. If the sequences $(X_n)$, $(\lambda_n)$ and $(p_n)$ satisfy the conditions
$$\lambda_m X_m = O(1) \quad \text{as} \quad m \to \infty,$$
$$\sum_{n=1}^{m} n X_n \Delta^2 \lambda_n = O(1) \quad \text{as} \quad m \to \infty,$$
$$\sum_{n=1}^{m} \frac{P_n}{p_n} |t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty,$$

then the series $\sum a_n \lambda_n$ is summable $|\tilde{N}, p_n|_k$, $k \geq 1$.

**Theorem 2.2.** [4] Let $(X_n)$ be a positive monotonic nondecreasing sequence. If the sequences $(X_n)$, $(\lambda_n)$, and $(p_n)$ satisfy the conditions (2.1)–(2.3) and
$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty,$$

then the series $\sum a_n \lambda_n$ is summable $|\tilde{N}, p_n|_k$, $k \geq 1$.

**Remark 2.1.** It should be noted that condition (2.5) is reduced to the condition (2.4), when $k = 1$. When $k > 1$, condition (2.5) is weaker than condition (2.4) but the converse is not true (see [4] for details).

3. An application of absolute matrix summability to Fourier series

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then $A$ defines the sequence-to-sequence transformation, mapping
the sequence \( s = (s_n) \) to \( A_s = (A_n(s)) \), where \( A_n(s) = \sum_{n=0}^{\infty} a_n s_n, n = 0, 1, \ldots \)

The series \( \sum a_n \) is said to be summable \( |A|_k, k \geq 1 \), if (see [13])

\[
\sum_{n=1}^{\infty} n^{k-1}|\Delta A_n(s)|^k < \infty,
\]

and it is said to be summable \( |A,p_n|_k, k \geq 1 \), if (see [12])

\[
\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{k-1} |\Delta A_n(s)|^k < \infty.
\]

where \( \Delta A_n(s) = A_n(s) - A_{n-1}(s) \).

If we take \( p_n = 1 \) for all \( n \), then \( |A,p_n|_k \) summability is the same as \( |A|_k \) summability. Also, if we take \( a_n = \frac{\alpha}{n^\alpha} \), then \( |A,p_n|_k \) summability is the same as \( |\tilde{N},p_n|_k \) summability. For any sequence \((\lambda_n)\) we write \( \Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1} \) and \( \Delta \lambda_n = \lambda_n - \lambda_{n+1} \). A sequence \((\lambda_n)\) is said to be of bounded variation, denoted by \((\lambda_n) \in BV\), if \( \sum_{n=1}^{\infty} |\Delta \lambda_n| < \infty \). Let \( f(t) \) be a periodic function with period \( 2\pi \), and Lebesgue integrable over \((-\pi, \pi)\). Write

\[
f(x) \sim \frac{1}{2} a_0 + \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} C_n(x),
\]

\( \phi(t) = \frac{1}{2}[f(x + t) + f(x - t)] \), and \( \phi_n(t) = \frac{\alpha}{\pi^\alpha} \int_0^\pi (t - u)^{\alpha-1}\phi(u) \, du \quad (\alpha > 0) \).

It is well known that if \( \phi(t) \in BV(0, \pi) \), then \( t_n(x) = O(1) \), where \( t_n(x) \) is the \((C, 1)\) mean of the sequence \((nC_n(x))\) (see [7]). Many works have been done dealing with absolute summability factors of Fourier series (see [3],[5],[11]). Among them, in [4], Bor has proved the following theorem dealing with the Fourier series.

**Theorem 3.1.** If \( \phi_1(t) \in BV(0, \pi), (X_n) \) is a positive monotonic nondecreasing sequence, the sequences \((p_n), (\lambda_n)\) satisfy conditions (2.1)–(2.3) and

\[
\sum_{n=1}^{m} \frac{p_n}{P_n} \left| \frac{t_n(x)}{x_n} \right|^k = O(X_m) \quad \text{as} \quad m \to \infty,
\]

then the series \( \sum C_n(x) \lambda_n \) is summable \( |\tilde{N},p_n|_k, k \geq 1 \).

If we take \( p_n = 1 \) for all values of \( n \), then we obtain a new result dealing with \( |C, 1|_k \) summability factors of Fourier series.

**4. Main Results**

We generalize Theorem 3.1 for \( |A,p_n|_k \) summability factors of Fourier series. Before stating the main theorem, we must first introduce some further notations.

With a normal matrix \( A = (a_{nv}) \), we associate two lower semimatrices \( \tilde{A} = (\tilde{a}_{nv}) \) and \( \bar{A} = (\bar{a}_{nv}) \) where \( \tilde{a}_{nv} = \sum_{i=v}^{n} a_{ni}, n, v = 0, 1, \ldots \) and \( \bar{a}_{00} = a_{00} = a_{00} \).
\( \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \ n = 1, 2, \ldots \) We note that \( \bar{A} \) and \( \bar{\hat{A}} \) are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. So, we have

\[
(4.1) \quad A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \bar{a}_{nv}a_v \quad \text{and} \quad \bar{\Delta} A_n(s) = \sum_{v=0}^{n} \bar{a}_{nv}a_v.
\]

**Theorem 4.1.** Let \( k \geq 1 \) and \( A = (a_{nv}) \) be a positive normal matrix such that

\[
\bar{a}_{n0} = 1, \ n = 0, 1, \ldots, \ a_{n-1,v} \geq a_{nv}, \ \text{for} \ n \geq v + 1,
\]

\[
a_{nn} = O\left(p_n/P_n\right), \ \bar{a}_{n,v+1} = O(v|\Delta_v(\bar{a}_{nv})|).
\]

If all the conditions of Theorem 3.1 are satisfied, then the series \( \sum C_n(x)\lambda_n \) is summable \( |A, p_n|_k, k \geq 1 \).

If we take \( a_{nv} = \frac{x^v}{n^v} \), then we get Theorem 3.1. We need the following lemma for the proof of our theorem.

**Lemma 4.1.** Under the conditions of Theorem 2.2 we have

\[
nX_n|\Delta\lambda_n| = O(1) \ \text{as} \ n \to \infty, \ \text{and} \ \sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty.
\]

**5. Proof of Theorem 4.1**

Let \( (I_n(x)) \) denote the \( A \)-transform of the series \( \sum_{n=1}^{\infty} C_n(x)\lambda_n \). Then, by (4.1), we have \( \bar{\Delta} I_n(x) = \sum_{v=1}^{n} \hat{a}_{nv}C_v(x)\lambda_v \). Applying Abel’s transformation to this sum, we get

\[
\bar{\Delta} I_n(x) = \sum_{v=1}^{n} \bar{a}_{nv}C_v(x)\lambda_v \frac{v}{v} = \sum_{v=1}^{n-1} \Delta_v \left( \frac{\hat{a}_{nv}\lambda_v}{v} \right) \sum_{r=1}^{v} rC_r(x) + \bar{\Delta} \lambda_n \sum_{r=1}^{n} rC_r(x)
\]

\[
= \sum_{v=1}^{n-1} \Delta_v \left( \frac{\hat{a}_{nv}\lambda_v}{v} \right) (v + 1)t_v(x) + \hat{a}_{nn}\lambda_n \frac{n+1}{n} t_n(x)
\]

\[
= \sum_{v=1}^{n-1} \Delta_v (\hat{a}_{nv})\lambda_v t_v(x) \frac{v+1}{v} + \sum_{v=1}^{n-1} \bar{a}_{n,v+1}\Delta\lambda_v t_v(x) \frac{v+1}{v}
\]

\[
+ \sum_{v=1}^{n-1} \bar{a}_{n,v+1}\lambda_{v+1} t_v(x) \frac{v}{v} + a_{nn}\lambda_n t_n(x) \frac{n+1}{n}
\]

\[
= I_{n,1}(x) + I_{n,2}(x) + I_{n,3}(x) + I_{n,4}(x).
\]

To complete the proof of Theorem 4.1 by Minkowski’s inequality, it is sufficient to show that

\[
\sum_{n=1}^{\infty} \left( \frac{p_n}{P_n} \right)^{k-1} |I_{n,r}(x)|^k < \infty, \ \text{for} \ r = 1, 2, 3, 4.
\]
First, by applying Hölder’s inequality with indices $k$ and $k'$, where $k > 1$ and \( \frac{1}{k} + \frac{1}{k'} = 1 \), we have

\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{k-1} |I_{n,1}(x)|^k \leq \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{k-1} \left\{ \sum_{v=1}^{n-1} \frac{v+1}{v} ||\Delta_v(\hat{a}_{nv})|||\lambda_v||t_v(x)| \right\}^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{k-1} \left\{ \sum_{v=1}^{n-1} ||\Delta_v(\hat{a}_{nv})|||\lambda_v||t_v(x)| \right\}^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{k-1} a_{mn}^{k-1} \left\{ \sum_{v=1}^{n-1} ||\Delta_v(\hat{a}_{nv})|||\lambda_v||t_v(x)| \right\}
\]

\[
= O(1) \sum_{v=1}^{m} |\lambda_v|^{k-1} |\lambda_v||t_v(x)|^k \sum_{n=v+1}^{m+1} ||\Delta_v(\hat{a}_{nv})||
\]

\[
= O(1) \sum_{v=1}^{m} |\lambda_v|^{k-1} |\lambda_v||t_v(x)|^k \sum_{n=v+1}^{m+1} X_v
\]

\[
= O(1) \sum_{v=1}^{m} X_v |\Delta_v| + O(1) |\lambda_m| X_m = O(1) \quad \text{as} \quad m \to \infty,
\]

by virtue of the hypotheses of Theorem 4.1 and Lemma 4.1. Now, using Hölder’s inequality we have

\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{k-1} |I_{n,2}(x)|^k \leq \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{k-1} \left\{ \sum_{v=1}^{n-1} \frac{v+1}{v} ||\Delta_v(\hat{a}_{nv})|||\lambda_v||t_v(x)| \right\}^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{k-1} \left\{ \sum_{v=1}^{n-1} ||\Delta_v(\hat{a}_{nv})|||\lambda_v||t_v(x)| \right\}
\]

\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{k-1} \sum_{v=1}^{n-1} \left( v ||\Delta_v(\hat{a}_{nv})|||\lambda_v||t_v(x)| \right)^k
\]

\[
\times \left\{ \sum_{v=1}^{n-1} ||\Delta_v(\hat{a}_{nv})|| \right\}^{k-1}
\]

\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{k-1} a_{mn}^{k-1} \sum_{v=1}^{n-1} \left( v ||\Delta_v(\hat{a}_{nv})|||\lambda_v||t_v(x)| \right)^k
\]

\[
= O(1) \sum_{v=1}^{m} \left( v ||\Delta_v(\hat{a}_{nv})|||\lambda_v||t_v(x)| \right)^k \sum_{n=v+1}^{m+1} ||\Delta_v(\hat{a}_{nv})||
\]
\[ \begin{align*}
&= O(1) \sum_{v=1}^{m} \frac{p_v}{P_v} \frac{1}{X_v^{k-1}} |t_v(x)|^k (v|\Delta \lambda_v|) \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v|\Delta \lambda_v|) \sum_{v=1}^{m} \frac{p_v}{P_v} \frac{1}{X_v^{k-1}} |t_v(x)|^k \\
&\quad + O(1)m|\Delta \lambda_m| \sum_{v=1}^{m} \frac{p_v}{P_v} \frac{1}{X_v^{k-1}} |t_v(x)|^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta(v|\Delta \lambda_v|)|X_v + O(1)m|\Delta \lambda_m|X_m \\
&= O(1) \sum_{v=1}^{m-1} vX_v|\Delta^2 \lambda_v| + O(1) \sum_{v=1}^{m-1} X_v|\Delta \lambda_v| + O(1)m|\Delta \lambda_m|X_m = O(1)
\end{align*} \]

as \( m \to \infty \), by virtue of the hypotheses of Theorem 4.1 and Lemma 4.1. Again, we have that

\[ \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{k-1} |f_{n,\alpha}(x)|^k = \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{k-1} \left| \sum_{v=1}^{n-1} \tilde{a}_{n,v+1} \lambda_{v+1} \frac{t_v(x)}{v} \right|^k \]

\[ \leq \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{k-1} \left\{ \sum_{v=1}^{n-1} |\Delta_v(\tilde{a}_{nv})||\lambda_{v+1}||t_v(x)| \right\}^k \]

\[ = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{k-1} \left\{ \sum_{v=1}^{n-1} |\Delta_v(\tilde{a}_{nv})||\lambda_{v+1}||t_v(x)| \right\}^k \]

\[ = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{k-1} \sum_{v=1}^{n-1} |\Delta_v(\tilde{a}_{nv})||\lambda_{v+1}||t_v(x)| \times \left\{ \sum_{v=1}^{n-1} |\Delta_v(\tilde{a}_{nv})| \right\}^{k-1} \]

\[ = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{k-1} \tilde{a}_{nn}^{k-1} \sum_{v=1}^{n-1} |\Delta_v(\tilde{a}_{nv})||\lambda_{v+1}||t_v(x)|^k \]

\[ = O(1) \sum_{v=1}^{m} |\lambda_{v+1}|^k |t_v(x)|^k \sum_{n=v+1}^{m+1} |\Delta_v(\tilde{a}_{nv})| \]

\[ = O(1) \sum_{v=1}^{m} \frac{p_v}{P_v} |t_v(x)|^k |\lambda_{v+1}|^{k-1} |\lambda_{v+1}| \]

\[ = O(1) \sum_{v=1}^{m} \frac{1}{X_v^{k-1}} |\lambda_{v+1}| |t_v(x)|^k \frac{p_v}{P_v} = O(1) \quad \text{as} \quad m \to \infty, \]
by virtue of the hypotheses of Theorem 4.1 and Lemma 4.1. Finally, as in $T_n$, we have that

$$
\sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{k-1} |I_{n,k}(x)| = O(1) \sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{k-1} a_{n,n} |\lambda_n|^k |I_n(x)|^k
$$

$$
= O(1) \sum_{n=1}^{m} \frac{p_n}{P_n} |\lambda_n|^k |I_n(x)|^k
$$

$$
= O(1) \sum_{n=1}^{m} \frac{1}{\lambda_n} |\lambda_n||I_n(x)|^k \frac{P_n}{p_n} = O(1) \text{ as } m \to \infty,
$$

by virtue of hypotheses of the Theorem 4.1 and Lemma 4.1. This completes the proof of Theorem 4.1.

If we take $a_{n,v} = \frac{p_v}{P_v}$ in Theorem 4.1 then we get Theorem 3.1 and if we take $p_n = 1$ for all values of $n$ in Theorem 4.1 then we get a new result dealing with the $|A|^k$ summability method. Also, if we take $a_{n,v} = \frac{p_v}{P_v}$ and $p_n = 1$ for all values of $n$ in Theorem 4.1 then we get a result concerning the $|C,1|^k$ summability methods.

References