Abstract. We study 3-dimensional $f$-Kenmotsu manifolds with the Schouten–van Kampen connection. With the help of such a connection, we study projectively flat, conharmonically flat, Ricci semisymmetric and semisymmetric 3-dimensional $f$-Kenmotsu manifolds. Finally, we give an example of 3-dimensional $f$-Kenmotsu manifolds with the Schouten–van Kampen connection.

1. Introduction

The Schouten–van Kampen connection is one of the most natural connections adapted to a pair of complementary distributions on a differentiable manifold endowed with an affine connection $^{[2]}$. Solov’ev investigated hyperdistributions in Riemannian manifolds using the Schouten–van Kampen connection $^{[12]}$. Then Olszak studied the Schouten–van Kampen connection to an almost contact metric structure $^{[8]}$. He characterized some classes of almost contact metric manifolds with the Schouten–van Kampen connection and found certain curvature properties of this connection on these manifolds.

On the other hand, let $M$ be an almost contact manifold, i.e., $M$ is a connected $(2n+1)$-dimensional differentiable manifold endowed with an almost contact metric structure $(\phi, \xi, \eta, g)$ $^{[1]}$. Denote by $\Phi$ the fundamental 2-form of $M$, $\Phi(X,Y) = g(X,\phi Y)$, $X,Y \in \chi(M)$, $\chi(M)$ being the Lie algebra of differentiable vector fields on $M$.

For further use, we recall the following definitions $^{[1,3,10]}$. The manifold $M$ and its structure $(\phi, \xi, \eta, g)$ is said to be:

i) normal, if the almost complex structure defined on the product manifold $M \times \mathbb{R}$ is integrable (equivalently $[\phi, \phi] + 2d\eta \otimes \xi = 0$),

ii) almost cosymplectic, if $d\eta = 0$ and $d\Phi = 0$.

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iii) cosymplectic, if it is normal and almost cosymplectic (equivalently, \( \nabla \phi = 0 \), \( \nabla \) being covariant differentiation with respect to the Levi-Civita connection).

The manifold \( M \) is called locally conformal, cosymplectic (respectively almost cosymplectic), if \( M \) has an open covering \( \{ U_t \} \) endowed with differentiable functions \( \sigma_t : U_t \to \mathbb{R} \) such that over each \( U_t \) the almost contact metric structure \( (\phi_t, \xi_t, \eta_t, g_t) \) defined by

\[
\phi_t = \phi, \quad \xi_t = e^{\sigma_t} \xi, \quad \eta_t = e^{-\sigma_t} \eta, \quad g_t = e^{-2\sigma_t} g
\]

is cosymplectic (respectively almost cosymplectic).

Also, Olszak and Rosca \[9\] studied normal locally conformal almost cosymplectic manifolds. They given a geometric interpretation of \( f \)-Kenmotsu manifolds and studied some curvature properties. Among others they proved that a Ricci symmetric \( f \)-Kenmotsu manifold is an Einstein manifold.

By an \( f \)-Kenmotsu manifold, we mean an almost contact metric manifold which is normal and locally conformal almost cosymplectic manifold.

In the present paper we study some curvature properties of 3-dimensional \( f \)-Kenmotsu manifolds with the Schouten–van Kampen connection. The paper is organized as follows: after introduction, we give the Schouten–van Kampen connection and \( f \)-Kenmotsu manifolds. Then we adapt the Schouten–van Kampen connection on 3-dimensional \( f \)-Kenmotsu manifolds. In section 5 we study projectively flat 3-dimensional \( f \)-Kenmotsu manifolds with the Schouten–van Kampen connection. In section 6 we consider conharmonically flat 3-dimensional \( f \)-Kenmotsu manifolds with the Schouten–van Kampen connection. Section 7 is devoted to study Ricci semisymmetric 3-dimensional \( f \)-Kenmotsu manifolds with the Schouten–van Kampen connection and we prove that if a 3-dimensional \( f \)-Kenmotsu manifold is Ricci semisymmetric, then it is an \( \eta \)-Einstein manifold. In section 8 we study semisymmetric 3-dimensional \( f \)-Kenmotsu manifolds with the Schouten–van Kampen connection. Finally, we give an example of a 3-dimensional \( f \)-Kenmotsu manifold with the Schouten–van Kampen connection which verifies Theorem 5.1 and Theorem 6.1.

2. The Schouten–van Kampen connection

Let \( M \) be a connected pseudo-Riemannian manifold of an arbitrary signature \((p, n - p), 0 \leq p \leq n, n = \dim M \geq 2 \). By \( g \) and \( \nabla \) we denote the pseudo-Riemannian metric and Levi-Civita connection induced from the metric \( g \) on \( M \) respectively. Assume that \( H \) and \( V \) are two complementary, orthogonal distributions on \( M \) such that \( \dim H = n - 1, \dim V = 1 \), and the distribution \( V \) is non-null. Thus \( TM = H \oplus V, H \cap V = \{ 0 \} \) and \( H \perp V \). Assume that \( \xi \) is a unit vector field and \( \eta \) is a linear form such that \( \eta(\xi) = 1, g(\xi, \xi) = \varepsilon = \pm 1 \) and

\[
H = \ker \eta, \quad V = \text{span}\{\xi\}.
\]

We can always choose such \( \xi \) and \( \eta \) at least locally (in a certain neighborhood of an arbitrarily chosen point of \( M \)). We also have \( \eta(X) = \varepsilon g(X, \xi) \). Moreover, it holds that \( \nabla_X \xi \in H \).
For any $X \in TM$, by $X^h$ and $X^v$ we denote the projections of $X$ onto $H$ and $V$, respectively. Thus, we have $X = X^h + X^v$ with

$$X^h = X - \eta(X)\xi, \quad X^v = \eta(X)\xi.$$  

(2.1)

The Schouten–van Kampen connection $\tilde{\nabla}$ associated to the Levi-Civita connection $\nabla$ and adapted to the pair of the distributions $(H, V)$ is defined by

$$\tilde{\nabla}_X Y = \left(\nabla_X Y^h\right)^h + \left(\nabla_X Y^v\right)^v,$$

(2.2)

and the corresponding second fundamental form $B$ is defined by $B = \nabla - \tilde{\nabla}$. Note that condition (2.2) implies the parallelism of the distributions $H$ and $V$ with respect to the Schouten–van Kampen connection $\tilde{\nabla}$.

From (2.1), one can compute

$$(\nabla_X Y^h)^h = \nabla_X Y - \eta(\nabla_X Y)\xi - \eta(Y)\nabla_X \xi,$$

$$(\nabla_X Y^v)^v = (\nabla_X \eta)(Y)\xi + \eta(\nabla_X Y)\xi,$$

which enables us to express the Schouten–van Kampen connection with help of the Levi-Civita connection in the following way

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi.$$  

(2.3)

Thus, the second fundamental form $B$ and the torsion $\tilde{T}$ of $\tilde{\nabla}$ are

$$B(X, Y) = \eta(Y)\nabla_X \xi - (\nabla_X \eta)(Y)\xi,$$

$$\tilde{T}(X, Y) = \eta(X)\nabla_Y \xi - \eta(Y)\nabla_X \xi + 2\eta(\nabla_X Y)\xi.$$  

With the help of the Schouten–van Kampen connection (2.3), many properties of some geometric objects connected with the distributions $H, V$ can be characterized. Probably, the most spectacular is the following statement: $g, \xi$ and $\eta$ are parallel with respect to $\tilde{\nabla}$, that is, $\tilde{\nabla} \xi = 0$, $\tilde{\nabla} g = 0$, $\tilde{\nabla} \eta = 0$.

3. $f$-Kemnotsu manifolds

Let $M$ be a real $(2n + 1)$-dimensional differentiable manifold endowed with an almost contact structure $(\phi, \xi, \eta, g)$ satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

(3.1)

$$\phi \xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi),$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X) \eta(Y),$$

for any vector fields $X, Y \in \chi(M)$, where $I$ is the identity of the tangent bundle $TM$, $\phi$ is a tensor field of $(1, 1)$-type, $\eta$ is a 1-form, $\xi$ is a vector field and $g$ is a metric tensor field. We say that $(M, \phi, \xi, \eta, g)$ is a $f$-Kemnotsu manifold if the Levi-Civita connection of $g$ satisfy

$$(\nabla_X \phi)(Y) = f \{g(\phi X, Y)\xi - \eta(Y)\phi X\},$$

where $f \in C^\infty(M)$ such that $df \wedge \eta = 0$. If $f = \alpha = \text{constant} \neq 0$, then the manifold is an $\alpha$-Kemnotsu manifold [7]. 1-Kemnotsu manifold is a Kemnotsu manifold [6]. If $f = 0$, then the manifold is cosymplectic [5]. An $f$-Kemnotsu manifold is said to be regular if $f^2 + f' \neq 0$, where $f' = \xi(f)$. 


For an $f$-Kenmotsu manifold from (3.1) it follows that
\begin{equation}
∇_X ξ = f\{X - η(X) ξ\}.
\end{equation}
Then using (3.2), we have
\begin{equation}
(∇_X η)(Y) = f\{g(X, Y) - η(X) η(Y)\}.
\end{equation}
The condition $df ∧ η = 0$ holds if $\dim M ≥ 5$. This does not hold in general if $\dim M = 3$.

As is well known, in a 3-dimensional Riemannian manifold, we always have
\begin{equation}
R(X, Y)Z = g(Y, Z) QX - g(X, Z) QY + S(Y, Z) X - S(X, Z) Y
\end{equation}
\[
-\frac{τ}{2}\{g(Y, Z) X - g(X, Z) Y\}.
\]
In a 3-dimensional $f$-Kenmotsu manifold $M$, we have [9]
\begin{equation}
R(X, Y)Z = \left(\frac{τ}{2} + 2f^2 + 2f'\right)\{g(Y, Z) X - g(X, Z) Y\}
\end{equation}
\[
-\left(\frac{τ}{2} + 3f^2 + 3f'\right)\{g(Y, Z) η(X) ξ - g(X, Z) η(Y) ξ + η(Y) η(Z) X - η(X) η(Z) Y\},
\end{equation}
\begin{equation}
S(X, Y) = \left(\frac{τ}{2} + f^2 + f'\right) g(X, Y) - \left(\frac{τ}{2} + 3f^2 + 3f'\right) η(X) η(Y),
\end{equation}
\begin{equation}
QX = \left(\frac{τ}{2} + f^2 + f'\right) X - \left(\frac{τ}{2} + 3f^2 + 3f'\right) η(X) ξ,
\end{equation}
where $R$ denotes the curvature tensor, $S$ is the Ricci tensor, $Q$ is the Ricci operator and $τ$ is the scalar curvature of $M$.

From (3.4) and (3.5), we obtain
\begin{equation}
R(X, Y)ξ = -(f^2 + f')\{η(Y)X - η(X)Y\},
\end{equation}
\begin{equation}
S(X, ξ) = -2(f^2 + f') η(X).
\end{equation}

4. 3-dimensional $f$-Kenmotsu manifolds with the Schouten–van Kampen connection

Let $M$ be a 3-dimensional $f$-Kenmotsu manifold with the Schouten–van Kampen connection. Then using (3.2) and (3.3) in (2.3), we get
\begin{equation}
\tilde{∇}_X Y = ∇_X Y + f\{g(X, Y) ξ - η(Y) X\}.
\end{equation}
Let $R$ and $\tilde{R}$ be the curvature tensors of the Levi-Civita connection $∇$ and the Schouten–van Kampen connection $\tilde{∇}$,
\begin{equation}
R(X, Y) = [∇_X, ∇_Y] - ∇_{[X,Y]}, \quad \tilde{R}(X, Y) = [\tilde{∇}_X, \tilde{∇}_Y] - \tilde{∇}_{[X,Y]}.
\end{equation}
Using (4.1), by direct calculations, we obtain the following formula connecting $R$ and $\tilde{R}$ on a 3-dimensional $f$-Kenmotsu manifold $M$,
\begin{equation}
\tilde{R}(X, Y)Z = R(X, Y)Z + f^2\{g(Y, Z) X - g(X, Z) Y\}
\end{equation}
\[
+ f'\{g(Y, Z) η(X) ξ - g(X, Z) η(Y) ξ + η(Y) η(Z) X - η(X) η(Z) Y\}.
\]
We will also consider the Riemann curvature $(0, 4)$-tensors $\tilde{R}, R$, the Ricci tensors $\tilde{S}, S$, the Ricci operators $\tilde{Q}, Q$ and the scalar curvatures $\tilde{\tau}, \tau$ of the connections $\tilde{\nabla}$ and $\nabla$ are given by

\begin{align}
\tilde{R}(X, Y, Z, W) &= R(X, Y, Z, W) + f^2 \{g(Y, Z) g(X, W) - g(X, Z) g(Y, W)\} \\
&\quad + f' \{g(Y, Z) \eta(X) \eta(W) - g(X, Z) \eta(Y) \eta(W) \}
+ g(X, W) \eta(Y) \eta(Z) - g(Y, W) \eta(X) \eta(Z)\},
\end{align}

\begin{align}
\tilde{S}(Y, Z) &= S(Y, Z) + (2f^2 + f') g(Y, Z) + f' \eta(Y) \eta(Z),
\end{align}

\begin{align}
\tilde{Q}X &= QX + (2f^2 + f')X + f' \eta(X) \xi, \\
\tilde{\tau} &= \tau + 6f^2 + 4f'.
\end{align}

respectively, where

\begin{align}
\tilde{R}(X, Y, Z, W) &= g(\tilde{R}(X, Y)Z, W) \quad \text{and} \quad R(X, Y, Z, W) = g(R(X, Y)Z, W).
\end{align}

5. Projectively flat 3-dimensional $f$-Kenmotsu manifolds with the Schouten–van Kampen connection

In this section, we study projectively flat 3-dimensional $f$-Kenmotsu manifolds with respect to the Schouten–van Kampen connection. In a 3-dimensional $f$-Kenmotsu manifold, the projective curvature tensor with respect to the Schouten–van Kampen connection is given by

\begin{align}
\tilde{P}(X, Y)Z &= \tilde{R}(X, Y)Z - \frac{1}{2} \{\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y\}.
\end{align}

If $\tilde{P} = 0$, then the manifold $M$ is called projectively flat manifold with respect to the Schouten–van Kampen connection.

Let $M$ be a projectively flat manifold with respect to the Schouten–van Kampen connection. From (5.1), we have

\begin{align}
\tilde{R}(X, Y)Z &= \frac{1}{2} \{\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y\}.
\end{align}

Using (4.3) and (4.4) in (5.2), we get

\begin{align}
g(R(X, Y)Z, W) + f^2 \{g(Y, Z) g(X, W) - g(X, Z) g(Y, W)\} \\
&\quad + f' \{g(Y, Z) \eta(X) \eta(W) - g(X, Z) \eta(Y) \eta(W) \}
+ g(X, W) \eta(Y) \eta(Z) - g(Y, W) \eta(X) \eta(Z)\}
&\quad = \frac{1}{2} \{S(Y, Z) g(X, W) - S(X, Z) g(Y, W) \\
&\quad + [2f^2 + f'][g(Y, Z) g(X, W) - g(X, Z) g(Y, W)] \\
&\quad + f' \eta(Y) \eta(Z) g(X, W) - \eta(X) \eta(Z) g(Y, W)\}.
\end{align}

Now putting $W = \xi$ in (5.3), we obtain

\begin{align}
(f^2 + f') \{g(Y, Z) \eta(Y) - g(Y, Z) \eta(X)\} + (f^2 + f') \{g(Y, Z) \eta(X) - g(X, Z) \eta(Y)\}
&\quad = \frac{1}{2} \{S(Y, Z) \eta(X) - S(X, Z) \eta(Y) + (2f^2 + f')[g(Y, Z) \eta(X) - g(X, Z) \eta(Y)]\},
\end{align}

\begin{align}
\tilde{S}(Y, Z) &= S(Y, Z) + (2f^2 + f') g(Y, Z) + f' \eta(Y) \eta(Z),
\end{align}

\begin{align}
\tilde{Q}X &= QX + (2f^2 + f')X + f' \eta(X) \xi, \\
\tilde{\tau} &= \tau + 6f^2 + 4f'.
\end{align}
which gives
\[ S(Y, Z) \eta(X) - S(X, Z) \eta(Y) + (2f^2 + f') [g(Y, Z) \eta(X) - g(X, Z) \eta(Y)] = 0. \]

Again putting \( X = \xi \) in (5.4), we get
\[ S(Y, Z) = -(2f^2 + f') g(Y, Z) - f' \eta(Y) \eta(Z). \]

Thus \( M \) is an \( \eta \)-Einstein manifold with respect to the Levi-Civita connection.

Also, using (5.5) in (4.4), we obtain
\[ \tilde{S}(Y, Z) = 0. \]

Hence the manifold \( M \) is a Ricci-flat manifold with respect to the Schouten–van Kampen connection. Then from (5.2) the manifold \( M \) is a flat manifold with respect to the Schouten–van Kampen connection.

Conversely, let \( M \) be a flat manifold with respect to the Schouten–van Kampen connection. Then we say that the manifold \( M \) is a Ricci-flat manifold with respect to the Schouten–van Kampen connection. Hence from (5.1), we get
\[ \tilde{P}(X, Y) Z = 0, \]
that is, the manifold \( M \) is a projectively flat manifold with respect to the Schouten–van Kampen connection. Thus we have the following:

**Theorem 5.1.** Let \( M \) be a 3-dimensional \( f \)-Kenmotsu manifold with the Schouten–van Kampen connection. Then the following statements are equivalent:

i) \( M \) is projectively flat with respect to the Schouten–van Kampen connection,

ii) \( M \) is Ricci flat with respect to the Schouten–van Kampen connection,

iii) \( M \) is flat with respect to the Schouten–van Kampen connection.

### 6. Conharmonically flat 3-dimensional \( f \)-Kenmotsu manifolds with the Schouten–van Kampen connection

In this section, we study conharmonically flat 3-dimensional \( f \)-Kenmotsu manifolds with respect to the Schouten–van Kampen connection. In a 3-dimensional \( f \)-Kenmotsu manifold the conharmonic curvature tensor with respect to the Schouten–van Kampen connection is given by

\[ \tilde{K}(X, Y) Z = \tilde{R}(X, Y) Z - \{ \tilde{S}(Y, Z) X - \tilde{S}(X, Z) Y + g(Y, Z) \tilde{Q} X - g(X, Z) \tilde{Q} Y \}. \]

If \( \tilde{K} = 0 \), then the manifold \( M \) is called **conharmonically flat** manifold with respect to the Schouten–van Kampen connection.

Let \( M \) be a conharmonically flat manifold with respect to the Schouten–van Kampen connection. From (6.1), we have

\[ \tilde{R}(X, Y) Z = \tilde{S}(Y, Z) X - \tilde{S}(X, Z) Y + g(Y, Z) \tilde{Q} X - g(X, Z) \tilde{Q} Y. \]

Using (4.3), (4.4) and (4.5) in (6.2), we get

\[ R(X, Y) Z + f^2 \{ g(Y, Z) X - g(X, Z) Y \} \]

\[ + f' \{ g(Y, Z) \eta(X) \xi - g(X, Z) \eta(Y) \xi + \eta(Y) \eta(Z) X - \eta(X) \eta(Z) Y \} \]

\[ = S(Y, Z) X - S(X, Z) Y \]

\[ + \left( 4f^2 + 2f' + \frac{r}{2} + f^2 + f' \right) \{ g(Y, Z) X - g(X, Z) Y \}. \]
Now putting $X = \xi$ in (6.3), we obtain

\begin{align}
R(\xi, Y)Z &+ (f^2 + f')\{g(Y, Z)\xi - \eta(Z)Y\} \\
&= S(Y, Z)\xi - S(\xi, Z)Y \\
&+ \left(4f^2 + 2f' + \frac{\tau}{2} + f''\right)\{g(Y, Z)\xi - \eta(Z)Y\} \\
&+ f'\{\eta(Y)\eta(Z)\xi - \eta(\xi)Y\} \\
&+ \left(f' - \frac{\tau}{2} - 3f^2 - 3f'\right)\{g(Y, Z)\xi - \eta(Z)\eta(Y)\xi\}.
\end{align}

Using (3.4) and (3.7) in (6.4), we get

\begin{align}
S(Y, Z)\xi - S(\xi, Z)Y + \left(4f^2 + 2f' + \frac{\tau}{2} + f''\right)\{g(Y, Z)\xi - \eta(Z)Y\} \\
&+ f'\{\eta(Y)\eta(Z)\xi - \eta(\xi)Y\} \\
&+ \left(f' - \frac{\tau}{2} - 3f^2 - 3f'\right)\{g(Y, Z)\xi - \eta(Z)\eta(Y)\xi\} = 0.
\end{align}

Taking the inner product with $\xi$ in (6.5), we have

\begin{align}
S(Y, Z) + 2(f^2 + f')\eta(Y)\eta(Z) + (2f^2 + f')\{g(Y, Z) - \eta(Y)\eta(Z)\} = 0,
\end{align}

which gives

\begin{align}
S(Y, Z) = -(2f^2 + f')g(Y, Z) - f'\eta(Y)\eta(Z).
\end{align}

Thus $M$ is an $\eta$-Einstein manifold with respect to the Levi-Civita connection.

Using (6.6) in (6.4), we obtain $\tilde{S}(Y, Z) = 0$. Hence the manifold $M$ is a Ricci-flat manifold with respect to the Schouten–van Kampen connection. Then from (6.2), the manifold $M$ is a flat manifold with respect to the Schouten–van Kampen connection.

Conversely, let $M$ be a flat manifold with respect to the Schouten–van Kampen connection. Then we say that the manifold $M$ is a Ricci-flat manifold with respect to the Schouten–van Kampen connection. Hence from (6.1), we get $\tilde{K}(X, Y)Z = 0$. i.e., the manifold $M$ is a conharmonically flat manifold with respect to the Schouten–van Kampen connection. Thus we have the following:

**Theorem 6.1.** Let $M$ be a $3$-dimensional $f$-Kenmotsu manifold with the Schouten–van Kampen connection. Then the following statements are equivalent:

i) $M$ is conharmonically flat with respect to the Schouten–van Kampen connection, 

ii) $M$ is Ricci flat with respect to the Schouten–van Kampen connection, 

iii) $M$ is flat with respect to the Schouten–van Kampen connection.
7. Ricci semisymmetric 3-dimensional $f$-Kenmotsu manifolds with the Schouten–van Kampen connection

A $f$-Kenmotsu manifold with the Schouten–van Kampen connection is called Ricci semisymmetric if $\bar{R}(X,Y) \cdot \bar{S} = 0$, where $\bar{R}(X,Y)$ is treated as a derivation of the tensor algebra for any tangent vectors $X, Y$. Then

\begin{equation}
\tilde{S}(\bar{R}(X,Y)Z,W) + \tilde{S}(Z,\bar{R}(X,Y)W) = 0.
\end{equation}

Using (4.3) and (4.4) in (7.1), we get

\begin{align*}
S(R(X,Y)Z,W) + S(Z,R(X,Y)W) + f' \{ \eta(R(X,Y)Z) \eta(W) \\
+ f' \eta(R(X,Y)W) \eta(Z) \} + f^2 \{ S(X,W) g(Y,Z) - S(Y,W) g(X,Z) \\
+ S(X,Z) g(Y,W) - S(Y,Z) g(X,W) \} \\
- f'(f^2 + f') \{ g(Y,Z) \eta(X) \eta(W) - g(X,Z) \eta(Y) \eta(W) + g(Y,W) \eta(X) \eta(Z) \\
- g(X,W) \eta(Y) \eta(Z) \} + f' \{ S(X,W) \eta(Y) \eta(Z) - S(Y,W) \eta(X) \eta(Z) \\
+ S(X,Z) \eta(Y) \eta(W) - S(Y,Z) \eta(X) \eta(W) \} = 0.
\end{align*}

Let $M$ be Ricci semisymmetric with respect to the Levi-Civita connection. Then we have

\begin{equation}
f'(\eta(R(X,Y)Z) \eta(W) + f' \eta(R(X,Y)W) \eta(Z)) + f^2 \{ S(X,W) g(Y,Z) \\
- S(Y,W) g(X,Z) + S(X,Z) g(Y,W) - S(Y,Z) g(X,W) \} \\
f'(f^2 + f') \{ g(Y,Z) \eta(X) \eta(W) - g(X,Z) \eta(Y) \eta(W) + g(Y,W) \eta(X) \eta(Z) \\
- g(X,W) \eta(Y) \eta(Z) \} + f' \{ S(X,W) \eta(Y) \eta(Z) - S(Y,W) \eta(X) \eta(Z) \\
+ S(X,Z) \eta(Y) \eta(W) - S(Y,Z) \eta(X) \eta(W) \} = 0.
\end{equation}

Putting $W = \xi$ in (7.2), we obtain

\begin{align*}
f' \eta(R(X,Y)Z) + f^2 \{ S(X,\xi) g(Y,Z) - S(Y,\xi) g(X,Z) \\
+ S(X,Z) \eta(Y) - S(Y,Z) \eta(X) \} \\
f'(f^2 + f') \{ g(Y,Z) \eta(X) - g(X,Z) \eta(Y) \} + f' \{ S(X,\xi) \eta(Y) \eta(Z) \\
- S(Y,\xi) \eta(X) \eta(Z) + S(X,Z) \eta(Y) - S(Y,Z) \eta(X) \} = 0.
\end{align*}

After some calculations, we get

\begin{equation}
2(f^2 + f')^2 \{ g(Y,Z) \eta(X) - g(X,Z) \eta(Y) \} \\
- (f^2 + f') \{ S(Y,Z) \eta(X) - S(X,Z) \eta(Y) \} = 0.
\end{equation}

Again putting $X = \xi$ in (7.3), we have

\begin{equation}
2(f^2 + f')^2 \{ g(Y,Z) - \eta(Y) \eta(Z) \} - (f^2 + f') \{ S(Y,Z) + 2(f^2 + f') \eta(Y) \eta(Z) \} = 0,
\end{equation}

which gives

\begin{equation}
(f^2 + f') \{ S(Y,Z) + 4(f^2 + f') \eta(Y) \eta(Z) - 2(f^2 + f') g(Y,Z) \} = 0.
\end{equation}
Let \( f^2 + f' \neq 0 \), then from (7.4), we get
\[
S(Y, Z) = 2(f^2 + f') g(Y, Z) - 4(f^2 + f') \eta(Y) \eta(Z).
\]
Hence the manifold is an \( \eta \)-Einstein manifold with respect to the Levi-Civita connection.

Using (7.5) in (4.4), we obtain
\[
\tilde{S}(Y, Z) = (4f^2 + 3f') g(Y, Z) - (4f^2 + 3f') \eta(Y) \eta(Z).
\]
Thus we have the following:

**Theorem 7.1.** Let \( M \) be a Ricci semisymmetric 3-dimensional regular \( f \)-Kenmotsu manifold with the Schouten–van Kampen connection. If \( M \) is a Ricci semisymmetric 3-dimensional \( f \)-Kenmotsu manifold with respect to the Levi-Civita connection, then \( M \) is an \( \eta \)-Einstein manifold with respect to the Schouten–van Kampen connection.

8. **Semisymmetric 3-dimensional \( f \)-Kenmotsu manifolds with the Schouten–van Kampen connection**

In this section, we study a semisymmetric regular 3-dimensional \( f \)-Kenmotsu manifold with the Schouten–van Kampen connection. If a 3-dimensional \( f \)-Kenmotsu manifold with the Schouten–van Kampen connection is *semisymmetric* then we can write
\[
(\tilde{R}(X, Y) \cdot \tilde{R})(Z, U)W = 0,
\]
which gives
\[
\tilde{R}(X, Y)\tilde{R}(Z, U)W - \tilde{R}(\tilde{R}(X, Y)Z, U)W
- \tilde{R}(Z, \tilde{R}(X, Y)U)W - \tilde{R}(Z, U)\tilde{R}(X, Y)W = 0.
\]
Using (4.2) in (8.1), we have
\[
\tilde{R}(X, Y)R(Z, U)W - R(\tilde{R}(X, Y)Z, U)W
- R(Z, \tilde{R}(X, Y)U)W - R(Z, U)\tilde{R}(X, Y)W = 0,
\]
which gives
\[
(\tilde{R}(X, Y) \cdot R)(Z, U)W = 0.
\]
Again using (4.2) in (8.2), we obtain
\[
- R(Z, U)R(X, Y)W + g(X, Z)R(Y, U)W - g(Y, U)R(Z, X)W
+ g(X, U)R(Z, Y)W - g(Y, W)R(Z, U)X + g(X, W)R(Z, U)Y
+ f' \{ g(R(Z, U)W, Y) \eta(X) \xi - g(R(Z, U)W, X) \eta(Y) \xi + \eta(R(Z, U)W) \eta(Y)X
\eta(Y) \eta(Z)R(X, U)W + g(Y, U) \eta(R(Z, X)W) \xi + \eta(Y) \eta(Z)R(X, U)W - g(Y, U) \eta(R(Z, X)W) \xi
\]

+ g(X, U) \eta(R(Z, Y)W)\xi - \eta(Y) \eta(U)R(Z, X)W + \eta(X) \eta(U)R(Z, Y)W
- g(Y, W) \eta(R(Z, U)X)\xi + g(X, W) \eta(R(Z, U)Y)\xi
- \eta(Y) \eta(W)R(Z, U)X + \eta(X) \eta(W)R(Z, U)Y \} = 0.

Now from (8.3), we can say:

If \(0 \neq f = \) constant (say \( f = \alpha \)), then \( f' = 0 \). Hence we get \( R \cdot R = -\alpha^2 Q(g, R) \).

Therefore the manifold \( M \) is a pseudosymmetric \( \alpha \)-Kenmotsu manifold.

If \( f \) is not constant, then using \( X = \xi \) in (8.3), we get

\[
\begin{align*}
(8.4) \quad R(\xi, Y)R(Z, U)W - R(R(\xi, Y)Z, U)W - R(Z, R(\xi, Y)U)W \\
- R(Z, U)R(\xi, Y)W + f^2 \{g(R(Z, U)W, Y)\xi - g(R(Z, U)W, \xi)Y \\
- g(Y, Z)R(\xi, U)W + g(\xi, Z)R(Y, U)W - g(Y, U)R(Z, \xi)W \\
+ g(\xi, U)R(Z, Y)W - g(Y, W)R(Z, U)\xi + g(\xi, W)R(Z, U)Y \} \\
+ f'(g(R(Z, U)W, Y)\xi - g(R(Z, U)W, \xi)Y + \eta(R(Z, U)W)Y)\xi \\
- \eta(R(Z, U)W)Y - g(Y, Z)\eta(R(\xi, U)W)\xi + g(\xi, Z)\eta(R(Y, U)W)\xi \\
- \eta(Y) \eta(Z)R(\xi, U)W + \eta(Z)R(Y, U)W - g(Y, U)\eta(R(Z, \xi)W)\xi \\
+ g(\xi, U)\eta(R(Z, Y)W)\xi - \eta(Y) \eta(U)R(Z, \xi)W + \eta(U)R(Z, Y)W \\
- g(Y, W)\eta(R(Z, U)\xi) + g(\xi, W)\eta(R(Z, U)Y) \\
- \eta(Y) \eta(W)R(Z, U)\xi + \eta(W)R(Z, U)Y \} = 0.
\end{align*}
\]

Taking the inner product with \( \xi \) in (8.4), we obtain

\[
(8.5) \quad \eta(R(\xi, Y)R(Z, U)W) - \eta(R(R(\xi, Y)Z, U)W) - \eta(R(Z, R(\xi, Y)U)W) \\
- \eta(R(Z, U)R(\xi, Y)W) + f^2 \{g(R(Z, U)W, Y) - g(R(Z, U)W, \xi)Y \\
- g(Y, Z)\eta(R(\xi, U)W) + g(\xi, Z)\eta(R(Y, U)W) - g(Y, U)\eta(R(Z, \xi)W) \\
+ g(\xi, U)\eta(R(Z, Y)W) - g(Y, W)\eta(R(Z, U)\xi) + g(\xi, W)\eta(R(Z, U)Y) \} \\
+ f'(g(R(Z, U)W, Y) - g(R(Z, U)W, \xi)Y + \eta(R(Z, U)W)Y) \\
- \eta(R(Z, U)W)Y - g(Y, Z)\eta(R(\xi, U)W) + g(\xi, Z)\eta(R(Y, U)W) \\
- \eta(Y) \eta(Z)\eta(R(\xi, U)W) + \eta(Z)\eta(R(Y, U)W) - g(Y, U)\eta(R(Z, \xi)W) \\
+ g(\xi, U)\eta(R(Z, Y)W) - \eta(Y) \eta(U)\eta(R(Z, \xi)W) + \eta(U)\eta(R(Z, Y)W) \\
- g(Y, W)\eta(R(Z, U)\xi) + g(\xi, W)\eta(R(Z, U)Y) \\
- \eta(Y) \eta(W)\eta(R(Z, U)\xi) + \eta(W)\eta(R(Z, U)Y) \} = 0.
\]

Let \( \{e_i\} \ (1 \leq i \leq 3) \) be an orthonormal basis of the tangent space at any point of \( M \). Then the sum for \( 1 \leq i \leq 3 \) of the relation (8.5) for \( Y = Z = e_i \) gives

\[
\begin{align*}
\eta(R(\xi, e_i)R(e_i, U)W) - \eta(R(R(\xi, e_i) e_i, U)W) - \eta(R(e_i, R(\xi, e_i)U)W) \\
- \eta(R(e_i, U)R(\xi, e_i)W) + f^2 \{g(R(e_i, U)W, e_i) - g(R(e_i, U)W, \xi)\eta(e_i) \\
- g(e_i, e_i)\eta(R(\xi, U)W) + g(\xi, e_i)\eta(R(e_i, U)W) - g(e_i, U)\eta(R(e_i, \xi)W) \\
+ g(\xi, U)\eta(R(e_i, e_i)W) - g(e_i, W)\eta(R(e_i, U)\xi) + g(\xi, W)\eta(R(e_i, U)e_i) \}
\end{align*}
\]
+ f′{(g(R(e_i, U)W, e_i) − g(R(e_i, U)W, ξ) η(e_i) + η(R(e_i, U)W) η(e_i) − η(R(e_i, U)W) η(e_i) − g(e_i, e_i) η(R(ξ, U)W) + g(ξ, e_i) η(R(e_i, U)W) − η(e_i) η(R(ξ, U)W) + η(e_i) η(R(e_i, U)W) − g(e_i, U) η(R(e_i, ξ)W) + g(ξ, U) η(R(e_i, e_i)W) − η(e_i) η(U) η(R(e_i, ξ)W) + η(U) η(R(e_i, e_i)W) − g(e_i, W) η(R(e_i, U)ξ) + g(ξ, W) η(R(e_i, U)e_i) − η(e_i) η(W) η(R(e_i, U)ξ) + η(W) η(R(e_i, U)e_i)} = 0.

After some calculations, we obtain

\[2(f^2 + f′)S(U, W) - 2g(R(ξ, W)U, ξ)\]

\[-f^2S(U, W) - 2g(R(ξ, W)U, ξ) - 2(f^2 + f′) η(U) η(W)\]

which gives

\[(f^2 + f′)S(U, W) - 2g(R(ξ, W)U, ξ) + 2(f^2 + f′) η(U) η(W)\] = 0.

Let f^2 + f′ ≠ 0. Then from (8.6), we get

\[(f^2 + f′)S(U, W) - 2g(R(ξ, W)U, ξ) + 2(f^2 + f′) η(U) η(W) = 0.\]

Using (3.6) in (8.7), we obtain S(U, W) = −2(f^2 + f′)g(U, W).

Thus we have the following:

**Theorem 8.1.** Let M be a 3-dimensional regular f-Kenmotsu manifold with
the Schouten–van Kampen connection. If M is semisymmetric with respect to the
Schouten–van Kampen connection, then:

i) If 0 ≠ f = α = constant, then the manifold M is a pseudosymmetric
α-Kenmotsu manifold, or,

ii) If f is not constant, then the manifold M is an Einstein manifold.

9. An example of a 3-dimensional f-Kenmotsu manifold
with the Schouten–van Kampen connection

We consider the 3-dimensional manifold M = \{(x, y, z) ∈ \mathbb{R}^3, z ≠ 0\}, where
(x, y, z) are the standard coordinates in \mathbb{R}^3. The vector fields

\[e_1 = z^2 \frac{∂}{∂x}, \quad e_2 = z^2 \frac{∂}{∂y}, \quad e_3 = \frac{∂}{∂z}\]

are linearly independent at each point of M. Let g be the Riemannian metric
defined by

\[g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.\]

Let η be the 1-form defined by η(Z) = g(Z, e_3) for any Z ∈ \chi(M). Let φ be the
(1,1) tensor field defined by φ(e_1) = −e_2, φ(e_2) = e_1, φ(e_3) = 0. Then using
linearity of φ and g we have

\[η(e_3) = 1, \quad φ^2Z = −Z + η(Z)e_3, \quad g(φZ, φW) = g(Z, W) − η(Z) η(W),\]
for any $Z,W \in \chi(M)$. Now, by direct computations we obtain

\[ [e_1, e_2] = 0, \quad [e_2, e_3] = -\frac{2}{z} e_2, \quad [e_1, e_3] = -\frac{2}{z} e_1. \]

The Riemannian connection $\nabla$ of the metric tensor $g$ is given by Koszul’s formula which is

\[
\begin{align*}
2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\
&\quad - g([X, Y], Z) - g(Y, [X, Z]) + g(Z, [X, Y]).
\end{align*}
\]

Using (9.1), we have

\[
2g(\nabla_{e_1} e_3, e_1) = 2g\left(-\frac{2}{z} e_1, e_1\right), \quad 2g(\nabla_{e_1} e_3, e_2) = 0 \quad \text{and} \quad 2g(\nabla_{e_1} e_3, e_3) = 0.
\]

Hence $\nabla_{e_1} e_3 = -\frac{2}{z} e_1$. Similarly, $\nabla_{e_2} e_3 = -\frac{2}{z} e_2$ and $\nabla_{e_3} e_3 = 0$. (9.1) further yields

\[
\begin{align*}
\nabla_{e_1} e_2 &= 0, & \nabla_{e_2} e_2 &= \frac{2}{z} e_3, & \nabla_{e_3} e_2 &= 0, \\
\nabla_{e_1} e_3 &= \frac{2}{z} e_3, & \nabla_{e_2} e_3 &= 0, & \nabla_{e_3} e_3 &= 0.
\end{align*}
\]

From (9.2), we see that the manifold satisfies $\nabla_X \xi = f\{X - \eta(X)\xi\}$ for $\xi = e_3$, where $f = -\frac{2}{z}$. Hence we conclude that $M$ is an $f$-Kenmotsu manifold. Also $f^2 + f' \neq 0$. Hence $M$ is a regular $f$-Kenmotsu manifold. \[\Box\]

It is known that

\[
R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.
\]

With the help of the above formula and using (9.3), it can be easily verified that

\[
\begin{align*}
R(e_1, e_2) e_3 &= 0, \quad R(e_2, e_3) e_3 = -\frac{6}{z^2} e_2, \\
R(e_1, e_3) e_3 &= -\frac{6}{z^2} e_1, \quad R(e_1, e_2) e_2 = -\frac{4}{z^2} e_1, \\
R(e_3, e_2) e_2 &= -\frac{6}{z^2} e_3, \quad R(e_1, e_3) e_2 = 0, \\
R(e_1, e_2) e_1 &= \frac{4}{z^2} e_2, \quad R(e_2, e_3) e_1 = 0, \\
R(e_1, e_3) e_1 &= \frac{6}{z^2} e_3.
\end{align*}
\]

Now the Schouten–van Kampen connection on $M$ is given by

\[
\begin{align*}
\hat{\nabla}_{e_1} e_3 &= \left(-\frac{2}{z} - f\right) e_1, \quad \hat{\nabla}_{e_2} e_3 = \left(-\frac{2}{z} - f\right) e_2, \\
\hat{\nabla}_{e_1} e_2 &= -f(e_1 - \xi), \quad \hat{\nabla}_{e_2} e_2 = 0, \\
\hat{\nabla}_{e_3} e_2 &= \frac{2}{z}(e_3 - \xi), \quad \hat{\nabla}_{e_3} e_2 = 0, \\
\hat{\nabla}_{e_1} e_1 &= \frac{2}{z}(e_1 - \xi), \quad \hat{\nabla}_{e_2} e_1 = 0, \\
\hat{\nabla}_{e_3} e_1 &= 0.
\end{align*}
\]
From (9.5), we can see that \( \tilde{\nabla}_e e_j = 0 \) (\( 1 \leq i, j \leq 3 \)) for \( \xi = e_3 \) and \( f = -\frac{2}{z} \). Hence \( M \) is a 3-dimensional \( f \)-Kenmotsu manifold with respect to the Schouten–van Kampen connection. Also using (9.4), it can be seen that \( \tilde{R} = 0 \). Thus the manifold \( M \) is a flat manifold with respect to the Schouten–van Kampen connection. Since a flat manifold is a Ricci-flat manifold with respect to the Schouten–van Kampen connection, the manifold \( M \) is both a projectively flat and a conharmonically flat 3-dimensional \( f \)-Kenmotsu manifold with respect to the Schouten–van Kampen connection. So, from Theorems 5.1 and 6.1, \( M \) is an \( \eta \)-Einstein manifold with respect to the Levi-Civita connection.

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References


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