ON THE CONVEXITY OF HIGH ORDER OF SEQUENCES

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Abstract. We improve some results of Lacković and Simić [2] concerning the weighted arithmetic means that preserve the convexity of high order of sequences.

In [1] and [3] a characterization is given for triangular matrices which define transformations in the set of sequences preserving convexity of order \( r \). In the particular case of weighted arithmetic means, explicit expressions were given before in Lacković and Simić [2]. In this paper we improve the results from [2] generalizing some of the properties that we proved in [7] for the convexity of order two.

At the beginning, let us specify some notation and definitions which will be used throughout the paper.

Let \( a = (a_n) (n = 0, 1, \ldots) \) be a real sequence. The \( r \)-th order difference of the sequence \( a \) is defined by:

\[
\Delta^r a_n = a_n - \Delta^{r-1} a_{n+1} - \Delta^{r-1} a_n \quad (r = 1, 2, \ldots; \ n + 0, 1, \ldots)
\]

Definition 1. A sequence \( a = (a_n) \) is said to be convex of order \( r \) if \( \Delta^r a_n \geq 0 \) for all \( n \in \mathbb{N} \).

Let \( p = (p_n) \) be a sequence of positive numbers. It defines a transformation \( P \) in the set of sequences: any sequence \( a = (a_n) \) is transformed into the sequence \( P(a) = A + (A_n) \) given by:

\[
A_n = \frac{p_0 a_0 + \cdots + p_n a_n}{p_0 + \cdots + p_n} \quad (a = 0, 1, \ldots)
\]

Definition 2. The Transformation \( P \) is said to be \( r \)-convex if the sequence \( A = P(a) \) is convex of order \( r \) for any sequence \( a \) convex of order \( r \).

In [2] the following theorem is given:

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**Theorem 0.** The transformation $P$ is $r$-convex if and only if the sequence $p = (p_n)$ is given by:

$$p_n = n! \cdot (p_0 + \cdots + p_{r-2}) \prod_{k=r-2}^{n-1} (k + 1)(p_0 + \cdots + p_{r-2}) + (r - 1)p_{r-1}$$

for $n \geq r$, with $p_0, \ldots, p_{r-1}$ arbitrary positive numbers.

**Remark.** For $a_0 = 0$ and $a_n = (3 + 6n - 2n^2)/3$ if $n \geq 1$, we have $\Delta^3 a_0 = 1$ and $\Delta^3 a_n = 0$ if $n \geq 1$, so that the sequence $(a_n)$ is convex of order 3. Let us choose for $r = 3$ : $p_0 = 6$, $p_1 = 1$ and $p_2 = 7/2$. From (3) we get $p_3 = 7/2$ and so from (2), we have $A_0 = 0$, $A_1 = 1/3, A_2 = 1$ and $A_3 = 1$, that is $\Delta^3 A_0 = -1$. Hence the result from Theorem 0 is not valid in this form. To amend it, we begin by putting (3) in a simpler shape. For this we use the following notation:

$$u \choose \alpha = 1, \quad \frac{u(u - 1) \cdots (u - n + 1)}{n!}, \quad \text{for } n \geq 1$$

where $u$ is an arbitrary real number.

**Lemma 1.** If the transformation $P$ is $r$-convex, then the sequence $(p_n)$ must be given by:

$$p_n = p_{r-1} \binom{u + n - 1}{n - r + 1} = \left( \frac{u}{n} \right), \quad \text{for } n \geq r$$

where

$$u = \frac{(r - 1) \cdot p_{r-1}}{p_0 + \cdots + p_{r-2}}, \quad p_k > 0 \quad \text{for } k = 0, \ldots, r - 1,$$

**Proof.** Because (5) is only a transcription of (3) using (4) and (6), the result was proved in [2]. However we sketch here another proof by mathematical induction. As in [2] we use the sequence $a_n = c \cdot n \cdot (n - 1) \cdots (n - r + 2)$ for which we have $\Delta^r a_n = 0$ for any $n$. Hence it is convex of order $r$ for any real $c$, and so must be $(A_n)$ too. But this happens if and only if for $c = 1$ we have $\Delta^r A_n = 0$ for any $n$. For $n = 0$ we get $p_r = p_{r-1}(u + r - 1)/r$ which is (5) for $n = r$. Suppose (5) is valid for $n \leq m$. To obtain $A_n$ for $r \leq n \leq m$, we must calculate:

$$\sum_{k=0}^{n} p_k = \sum_{k=0}^{r-2} p_k + p_{r-1} + \sum_{k=r}^{n} p_k = p_{r-1} \left[ \frac{r - 1}{u} + 1 + \sum_{i=0}^{n-r} \left( \binom{u + r + i - 1}{i + 1} \binom{r + i}{i + 1} \right) \right].$$

From this it can be shown, by mathematical induction, that:

$$\sum_{k=0}^{n} p_k = p_{r-1} \frac{n - r + 2}{u} \binom{u + n}{n - r + 2} = \binom{n}{n - r + 1} \cdot \frac{n}{n - r + 2}.$$ 

So:

$$A_n = \frac{u \cdot (r - 1)!}{u + r - 1} \binom{n}{r - 1}, \quad n \leq m$$
and
\[
A_{m+1} = \frac{p_r - 1}{u} \left( \binom{m}{r-1} \right) \cdot b_k
\]
From $\Delta^r A_{m+1} = 0$, we obtain (5) for $m + 1$, and so for every $n$.

**Lemma 2.** If the sequence $(a_n)$ is given by:

\[
a_n = \sum_{k=0}^{n} \binom{n + r - k - 1}{r - 1} \cdot b_k,
\]

then

\[
\Delta^r a_n = b_{n+r} \quad (n = 0, 1, \ldots)
\]

**Remark 2.** This result is connected with some relations from [1] and [6]. Because any sequence may be put in the form (8), we obtain a representation theorem simpler than that given in [6]:

**Corollary 1.** The sequence $(a_n)$ is convex of order $r$ if and only if in its representation (8), it has $b_n \geq 0$ for $n \geq r$.

**Lemma 3.** If the transformation $P$ is $r$-convex, then for every $n \leq r$:

\[
\sum_{k=0}^{n-1} p_k = n \cdot p_n/u.
\]

*Proof.* Let $(A_n)$ be represented by:

\[
A_n = \sum_{k=0}^{n} \binom{n + r - k - 1}{r - 1} \cdot c_k.
\]

Then:

\[
a_n = \left( A_n \sum_{i=0}^{n} p_i - A_{n-1} \sum_{i=0}^{n-1} p_i \right) : p_n.
\]

If

\[
q_n = \frac{1}{p_n} \sum_{k=0}^{n-1} p_k
\]

then

\[
\sigma_n = A_n + q_n \cdot (A_n - A_{n-1}) = \sum_{k=0}^{n} \left[ \binom{n + r - i - 1}{r - 1} + q_n \cdot \binom{n + r - i - 2}{r - 2} \right] c_i
\]
for \( n \geq 1 \) and \( a_0 = A_0 = c_0 \). So:

\[
\Delta^r a_0 = \sum_{j=0}^{r} (-1)^j \binom{r}{j} a_{r-j} = \\
= \sum_{j=0}^{r-1} \left\{ \sum_{i=0}^{r-1} \left[ \binom{2r-j-i-1}{r-1} + q_{r-j} \binom{2r-j-i-2}{r-2} \right] c_i \right\} (-1)^j \binom{r}{j} + (-1)^r c_0 \\
= \sum_{i=0}^{r} \left\{ \sum_{j=0}^{r-1} \left[ \binom{2r-j-i-1}{r-1} + q_{r-j} \binom{2r-j-i-2}{r-2} \right] (-1)^j \binom{r}{j} \right\} c_i + \\
+ \left\{ \sum_{j=0}^{r-1} \left[ \binom{2r-j-1}{r-1} + q_{r-j} \binom{2r-j-2}{r-2} \right] \right\} (-1)^j \binom{r}{j} + (-1)^r c_0.
\]

But, as it is proved in [5]:

\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} = 0 \text{ for } p < n
\]

and hence:

\[
\sum_{j=0}^{n} (-1)^j \binom{r}{j} \cdot Q(j) = 0
\]

for any polynomial \( Q \) of degree less than \( n \). So:

\[
(12) \quad \sum_{j=0}^{m} (-1)^j \binom{r}{j} \cdot \binom{m+r-j-1}{r-1} = 0 \text{ for } m = 1, \ldots, r
\]

because:

\[
\sum_{j=0}^{m} (-1)^j \frac{r!}{j! \cdot (r-j)!} \cdot \frac{(m+r-j-1)!}{(r-1)! \cdot (m-j)!} = \frac{r}{m} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \cdot \binom{m+r-j-1}{m-1}
\]

and \( \binom{m+r-j-1}{m-1} \) is a polynomial of deg \( m-1 \) in \( j \). Hence:

\[
\Delta^r a_0 = c_r + \sum_{i=0}^{r} \sum_{j=0}^{r-1} (-1)^j \binom{r}{j} \cdot \binom{2r-j-i-2}{r-2} \cdot q_{r-j} c_i + \\
+ \sum_{j=0}^{r-1} (-1)^j \binom{r}{j} \cdot \binom{2r-j-2}{r-2} \cdot q_{r-j} c_0.
\]

As the coefficient of \( c_r \) is \( 1 + q_r > 0 \), \( \Delta^r a_0 \geq 0 \) implies \( \Delta^r A_0 = c_r \geq 0 \) if and only if the coefficients of \( c_i \) are zero for \( i = 0, \ldots, r-1 \). For \( i = r-1 \) we have:

\[
(r-1) \cdot q_r - r \cdot q_{r-1} = 0 \text{ and as (6) means } q_{r-1} = (r-1)/u, \text{ we also have } q_r = r/u.
\]

Assuming (10) valid for \( r-j = 0, \ldots, m-1; m < r-1 \) it may be deduced for \( r-m \), because we have:

\[
\sum_{j=0}^{r-1} (-1)^j \binom{r}{j} \cdot \binom{m+r-j-2}{r-2} \cdot (r-j) = 0, \text{ for } m < r-1
\]
and
\[
\sum_{j=0}^{r-1} (-1)^j \binom{r}{j} \left( \frac{2r - j - 2}{r - 2} \right) (r - j) = 0
\]
which may be verified as in (12).

**Theorem 1.** The transformation \( P \) is \( r \)-convex if and only if the sequence \((p_n)\) is given by:
\[
p_n = p_0 \cdot \binom{u + n - 1}{n}, \quad \text{for } n \geq 1, \quad \text{with } u = p_1 / p_0.
\]

**Proof.** Necessity: Lemma 1 and Lemma 3 give the necessary conditions (5) and (10). From (10) we have: \( u = p_1 / p_0 \) for \( n = 1 \), and \( p_2 = u(p_0 + p_1)/2 = p_0 \binom{u+1}{2} \): supposing (13) valid for \( n \leq m < r - 1 \), (10) gives:
\[
p_{n+1} = \frac{u \cdot p_0}{m+1} \sum_{k=0}^{m} \binom{u+k-1}{k} = p_0 \frac{u}{m+1} \binom{u+m}{m} = p_0 \binom{u+m}{m+1}
\]
that is, (13) holds for \( n \leq r - 1 \). Hence, from (5) we also get:
\[
p_n = p_0 \cdot \binom{u + r - 2}{r - 1} \cdot \binom{u + n - 1}{n - r + 1} : \binom{n}{r - 1} = p_0 \cdot \binom{u + n - 1}{n}
\]
for \( n \geq r \).

Sufficiency: with (13), the sequence (2) becomes:
\[
A_n \left[ \sum_{k=0}^{n} \binom{u+k-1}{k} a_k \right] : \binom{u+n}{n}
\]
and so we have the relation:
\[
a_n = A_n + n \cdot (A_n - A_{n+1}) : u, \quad \text{for } n > 0.
\]

Taking \( A_n \) of the form (11), from (15) we obtain:
\[
a_n = \sum_{k=0}^{n} \binom{n+r-k-2}{r-2} \cdot \left( \frac{n+r-k-1}{r-1} + \frac{n}{u} \right) c_k.
\]

Because \( \Delta^r A_n = c_{n+r} \), applying to (15) the know relation (see [4]):
\[
\Delta^r (a_n \cdot b_n) = \sum_{i=0}^{r} \binom{r}{i} \Delta^i a_n \cdot \Delta^{r-i} b_{n+i}
\]
we obtain:
\[
\Delta^r a_n = (n+r+u)u^{-1} c_{n+r} - nu^{-1} c_{n+r-1}, \quad n \geq 1.
\]
From the proof of Lemma 3 we have: $\Delta^r a_0 = c_r \cdot (r + u) : u$, that is (17) is valid for $n = 0$ too. Assuming $(a_n)$ given by (8), (9) is valid; thus:

$$
(18) \quad b_r = (r + u)/u, \quad b_{n+r} = (n + r + u)/uc_{n+r} - n/u c_{n+r-1}.
$$

Hence, if $b_n \geq 0$ for $n \geq r$, then also $c_n \geq 0$ for $n \geq r$; that is, if $(a_n)$ is convex of order $r$, so is $(A_n)$ too.

\textbf{Remark 3.} The sufficiency part of Theorem 1 was also proved in [1]. In what follows we improve also this result. Let us denote by $K_r$, the set of all sequences convex of order $r$ and by $K_r^u$ the set of all sequences $(a_n)$ with the property that (14) gives a sequence $(A_n)$ in $K_r$.

\textbf{THEOREM 2.} If $0 < v < u$ then the following strict inclusions hold:

$$
K_r \subset K_r^u \subset K_r^v.
$$

\textbf{Proof.} The first inclusion was proved in Theorem 1. Its strictness follows from (18): the positivity of $c_n(n \geq r)$ does not imply that of $b_n$. Now suppose $(a_n)$ given by (16) and also by:

$$
a_n = \sum_{k=0}^{n} \binom{n + r - k - 2}{r - 2} \cdot \left( \frac{n + r - k - 1}{r - 1} + \frac{n}{v} \right) \cdot d_k.
$$

So (17) holds and $\Delta^r a_n = (n + r + v)u^{-1}d_{n+r} - nu^{-1}d_{n+r-1}$ that is:

$$
(n + r + v)/vd_{n+r} - nu^{-1}d_{n+r-1} = (n + r + u)u^{-1}c_{n+r} - nu^{-1}c_{n+r-1}
$$

Hence $d_r = \frac{u^{r+u}}{v^{r+v}} c_r$ and generally, by mathematical induction:

$$
d_{r+n} = \frac{u^{r+n} + v^{r+n}}{v^{r+n} + u^{r+n}} \cdot \frac{c_{r+n}}{uv} + \frac{u - v}{uv} \sum_{i=0}^{n-1} \frac{c_{r+i}}{n-i+1} \binom{n}{i} \cdot \left( \frac{v + r + n}{n - i + 1} \right);
$$

that is, $c_n \geq 0$ for $b \geq r$ implies $d_n \geq 0$ for $b \geq r$ and so, if $(a_n)$ is in $K_r^u$, it is also in $K_r^v$. That the inclusion $K_r^u \subset K_r^v$ is strict follows also from (19) as above.

\textbf{REFERENCES}


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