ON DISTANCES IN SOME BIPARTITE GRAPHS

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Abstract. Let \( d(v \mid G) \) be the sum of the distances between a vertex \( v \) of a graph \( G \) and all other vertices of \( G \). Let \( W(G) \) be the sum of the distances between all pairs of vertices of \( G \). A class \( \mathcal{C}_k \) of bipartite graphs is found, such that \( d(v \mid G) \equiv 1 \pmod{k} \) holds for an arbitrary vertex of an arbitrary member of \( \mathcal{C}(k) \). Further, for two members \( G \) and \( H \) of \( \mathcal{C}(k) \), having equal cyclomatic number, \( W(G) \equiv W(H) \pmod{2k^2} \).

Introduction

In the present paper we establish certain properties of the vertex distances of some bipartite graphs. If \( G \) is a (connected) graph and \( u \) and \( v \) are its two vertices, then the length of the shortest path which connects \( u \) and \( v \) is denoted by \( d(u, v) \) and is called the distance between \( u \) and \( v \). The sum of the distances between the vertex \( v \) and all other vertices of \( G \) is denoted by \( d(G \mid v) \). The sum of the distances between all pairs of vertices of \( G \) is denoted by \( W(G) \) or simply by \( W \). Hence,

\[
W = W(G) = \sum_{\{u, v\}} d(u, v)
\]

where \( \{u, v\} \) runs over all two-element subsets of the vertex set of \( G \).

We mention in passing that the quantity \( W \) plays some role in chemistry [1]. In the chemical literature \( W(G) \) is called the Wiener number of the graph \( G \).

Let \( G \) be a connected bipartite graph and \( X \) and \( Y \) its two pertinent vertex sets. Then one immediately sees that \( d(u, v) \) is even if both \( u \) and \( v \) belong to either \( X \) or to \( Y \). Otherwise, \( d(u, v) \) is odd. This implies the following consequence.

Lemma 1. \( d(v \mid G) \equiv 1 \pmod{2} \) iff either \( v \in X \) and \( |Y| \) is odd or \( v \in Y \) and \( |X| \) is odd. Further, \( W(G) \equiv 1 \pmod{2} \) iff both \( |X| \) and \( |Y| \) are odd.

In the present paper we prove a number of additional congruence statements for the numbers \( d(v \mid G) \) and \( W(G) \), which hold for the elements of the sets \( \mathcal{C}(h, k) \) and \( \mathcal{C}(k) \).

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**Definition.** Let \( k \) be a positive integer. If \( h > 1 \), then every element of \( C(h, k) \) is a graph obtained by joining the endpoints of a path with \( 2k \) vertices to a pair of adjacent vertices of some graph from \( C(h - 1, k) \). The set \( C(1, k) \) consists of one element only—the circuit with \( 2k + 2 \) vertices.

It is both consistent and convenient to define \( C(0, k) \) as the one-element set, containing graph on two vertices.

The union of the sets \( C(h, k), h = 0, 1, 2, \ldots \) is denoted by \( C(k) \).

For example, \( C(4, 1) \) consists of eight elements, namely the eight graphs depicted in Fig. 1.

![Graphs](image)

**Fig. 1**

The basic properties of the above defined classes of graphs are collected in the following lemma.

**Lemma 2.** If \( G \) is a graph from \( C(h, K) \), then
(a) \( G \) is a connected bipartite graph with \( |X| = |Y| \);
(b) the cyclomatic number of \( G \) is \( h \);
(c) the girth of \( G \) is \( 2k + 2 \) and every edge of \( G \) belongs to a \((2k + 2)\)-membered circuit;
(d) \( G \) has \( |G| = 2kh + 2 \) vertices.

**The main results**

**Theorem 1.** If graph from \( C(k) \) and \( v \) is its arbitrary vertex, then
\[
d(v \mid G) \equiv 1 \pmod{k},
\]
If, further, $k$ is even, then
\[
d(v \mid G) \equiv 1 \pmod{2k}.
\] (2)

Theorem 2. If $G$ and $H$ are graphs from $\mathcal{C}(h,k)$, then
\[
W(G) \equiv W(H) \pmod{2k^2}.
\] (3)

Proof of Theorem 1. We demonstrate the validity of Theorem 1 for $G \in \mathcal{C}(h,k)$ by induction on $h$. For $h = 0$, (1) and (2) hold in a trivial manner since then $W = 1$. For the unique graph from $\mathcal{C}(1,k)$, namely the circuit with $2k + 2$ vertices, it is easy to show that $d(v \mid G) = (k + 1)^2$. Whence (1) and (2) are satisfied.

Suppose now that $G^*$ is an element of $\mathcal{C}(h-1,k)$ and that $G$ can be obtained by joining the endpoints $u_1$ and $u_{2k}$ of a path with $2k$ vertices with the vertices $p$ and $q$ of $G^*$ (see Fig. 2).

![Diagram](image)

Fig. 2

The newly introduced vertices of $G$ are labeled by $u_1, u_2, \ldots, u_{2k}$.

From the construction of the graph $G$ it is evident that
\[
d(v \mid G) = d(v \mid G^*) + \sum_{i=1}^{2k} d(v, u_i).
\]

Assuming that $d(v, p) < d(v, q)$, we have
\[
\sum_{i=1}^{k} d(v, u_i) = kd(v, p) + k(k + 1)/2,
\]
\[
\sum_{i=k+1}^{2k} d(v, u_i) = k + kd(v, p) + k(k + 1)/2,
\]
and
\[
d(v \mid G) = d(v \mid G^*) + 2kd(v, p) + k(k + 2).
\]

Consequently,
\[
d(v \mid G) \equiv d(v \mid G^*) \pmod{k}
\]
and, if $k$ is even,
\[ d(v \mid G) \equiv d(v \mid G^*) \pmod{2k}. \]

Therefore if (1) and (2) hold for the vertex $v$ of $G^*$, then they also hold for the vertex $v$ of $G$.

In order to complete the proof of Theorem 1, we have to show that (1) and (2) hold also for the vertices $u_i, i = 1, 2, \ldots, 2k$ of $G$. Let $u$ stand for one of these vertices. Then
\[ d(u \mid G) = \sum_{i=1}^{2k} d(u, u_i) + d(u, p) + d(u, q) + \sum_v d(u, v) \quad (4) \]

with the second summation of the r.h.s. of (4) running over the vertices of $G^*$ different than $p$ and $q$. The vertices $u_1, u_2, \ldots, u_{2k}, q, p$ form a circuit of size $2k + 2$ in $G$ and therefore
\[ \sum_{i=1}^{2k} d(u, u_i) + d(u, p) + d(u, q) = k(k + 1)^2. \]

Further,
\[ \sum_v d(u, v) = d(x \mid G^*) - 1 + (|G^*| - 2)d(u, x) \]

where $x = p$ if $d(u, p) < d(u, q)$ and $x = q$ otherwise. According to Lemma 2, $|G^*| = 2k(h - 1) + 2$.

Taking all this into account, eq. (4) becomes
\[ d(u \mid G) = d(x \mid G^*) + k(k + 2) + 2k(h - 1)d(u, x), \]

Hence,
\[ d(u \mid G) \equiv d(x \mid G^*) \pmod{k} \]

and, if $k$ is even,
\[ d(u \mid G) \equiv d(x \mid G^*) \pmod{2k}. \]

This means that if (1) and (2) hold for all vertices of $G^* \in C(h - 1, k)$, then they also for all vertices of $G \in C(h, k)$. Consequently, they hold for all vertices of all graphs from $C(k)$. \hfill \square

**Proof of Theorem 2.** The sets $C(0, k), C(1, k)$ and $C(2, k)$ contain one element each and therefore for $h = 0, 1$ and $2$ Theorem 2 holds in a trivial manner. Direct calculation confirms the validity of Theorem 2 also in the case of $C(3, k)$ (which contains $k + 2$ elements).

In order to complete a proof by induction, assume that (3) is obeyed for all graphs from $C(h - 1, k)$ and in particular for $G^*$ and $H^*$. Let $G \in C(h, k)$ be obtained from $G^*$ in the previously described way (see Fig. 2). Let $H \in C(h, k)$ be obtained from $H^*$ in a fully analogous manner.
Now, from Fig. 2 we see that
\[ W(G) = W(G^*) + k(4k^2 - 1)/3 + k(k + 1) \mid G^* \mid + k[d(| G^* ) + d(q \mid G^*)]. \]  
(5)

Namely, \( k(4k^2 - 1)/3 \) is the \( W \) number of the path with \( 2k \) vertices, where as the sum of the distances between the vertices of \( G^* \) and \( u_1, u_2, \ldots, u_k \) is
\[ kd(p \mid G^*) \mid G^* \mid \sum_{i=1}^{k} i \]
and, similarly, the sum of the distances between the vertices of \( G^* \) and \( u_{k+1}, \ldots, u_k \) is
\[ kd(q \mid G^*) \mid G^* \mid \sum_{i=0}^{k} i. \]

An analogous equality will hold for the graph \( H \), viz.,
\[ W(H) = W(H^*) + k(4k^2 - 1)/3 + k(k + 1) \mid H^* \mid + \]
\[ k \mid d(p \mid H^*) + d(q \mid H^*) \mid . \]  
(6)

Bearing in mind that by Lemma 2, \( |G^*| = |H^*| \), the identities (5) and (6) yield
\[ W(G) - W(H) = W(G^*) - W(H^*) + k[d(p \mid G^*) - d(p \mid H^*) + d(q \mid G^*) - d(q \mid H^*)]. \]

We now have to distinguish between two cases.

Case a: \( k \) is even. Then because of Theorem 1, \( d(p \mid G^*) - d(p \mid H^*) + d(q \mid G^*) - d(q \mid H^*) \) is divisible by \( 2k \). Therefore the induction hypothesis that \( W(G^*) - W(H^*) \) is divisible by \( 2k^2 \) leads to the conclusion that then also \( W(G) - W(H) \) is divisible by \( 2k^3 \).

Theorem 2 follows for the case of even \( k \).

Case b: \( k \) is odd. Then by Theorem 1, \( d(p \mid G^*) - d(p \mid H^*) + d(q \mid G^*) - d(q \mid H^*) \) is divisible only by \( k \) and the above reasoning leads to the conclusion that \( W(G) - W(H) \) is divisible by \( k^3 \). On the other hand, according to Lemma 1, \( W(G) - W(H) \) must be divisible by two. Since \( k^2 \) is assumed to be odd, \( W(G) - W(H) \) must be divisible by \( 2k^3 \).

This proves Theorem 2 also for odd values of \( k \). □

Discussion

1° In the set \( C(3,k) \) there exist graphs \( G \) and \( H \) such that \( W(G) - W(H) = 2k^2 \). Therefore \( 2k^2 \) is the greatest possible argument in a relation of the type (3) and Theorem 2, is, in a certain sense, the best possible congruence statement for the \( W \) numbers of the members of \( C(h,k) \).

2° Eq. (5) and the fact that \( |G| = 2k(h - 1) + 2 \) imply
\[ W(G) - W(G^*) \equiv k(k^2 + 11)/3(\text{mod} \ k^3). \]
This means that by increasing the cyclomatic number by one, $W$ increases by $k(k^2 + 11)/3$ modulo $k^2$. Since for the (unique) element of $C(0,k)$ it is $W = 1$, one concludes that for the members of $C(h,k)$.

$$W \equiv 1 + hk(k^2 + 11)/3 \pmod{k^2}.$$  

Analogously, if $k$ is even, then

$$W \equiv 1 + hk(k^2 + 11)/3 \pmod{2k^2}.$$  

3° There is a natural way to generalize the definition of the presently considered bipartite graphs to non-bipartite ones. Instead of the set $C(h,k)$ we may define that the set $C^*(h,k)$ whose elements are constructed by joining the endpoints of a path with $2k - 1$ vertices to a pair adjacent vertices of a graph from $C^*(h-1,k)$. Further, $C^*(1,k)$ would consist of the circuit of the size $2k + 1$.

Unfortunately, neither Theorem 1 nor Theorem 2 could be extended to $C^*(h,k)$, nor any other similar congruence statement could be established.

REFERENCES


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