SOME SPECIAL CASES OF PARALLEL DISPLACEMENTS
IN RECURRENT FINSLER SPACES

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Abstract. Some special cycles of line elements in the recurrent Finsler space $F_n$ are considered. If the vector is parallelly transported along one of the cycles of line elements the difference between the original vector and the one obtained after parallel transportation is expressed by some of the curvature tensor. The method used here is the generalisation of that, used by Varga [1], for the non-recurrent Finsler space.

1. Introduction. Let us consider Finsler space $F_n$ in which the metric function is $F(x, \dot{x})$ and the metric tensor is defined by

$$g_{\alpha\beta}(x, \dot{x}) = 2^{-1} \partial_{\alpha} \partial_{\beta} F^2(x, \dot{x}).$$

Definition 1.1. The Finsler space is called recurrent and is denoted by $\overline{F}_n$ when there exist vector fields $\lambda_\gamma(x, \dot{x})$ and $\mu_\gamma(x, \dot{x})$ homogeneous of degree zero in $\dot{x}$ such that [2]

\begin{equation}
  g_{\alpha\beta}|_{\gamma} = \partial_\gamma g_{\alpha\beta} - F \partial_\beta g_{\alpha\beta} \Gamma^{\delta}_{\alpha \gamma} - \Gamma^{\alpha}_{\beta \gamma} g_{\delta\beta} - \Gamma^{\alpha}_{\beta \delta} g_{\alpha\gamma} = \lambda_\gamma g_{\alpha\beta}
\end{equation}

\begin{equation}
  g_{\alpha\beta}|_{\gamma} = F \partial_\beta g_{\alpha\beta} (\delta^\delta_{\gamma} - A^\delta_{\alpha \gamma}) - A^\delta_{\alpha \gamma} g_{\delta\beta} - A^\delta_{\beta \gamma} g_{\delta\beta} = \mu_\gamma g_{\alpha\beta}
\end{equation}

\begin{equation}
  D g_{\alpha\beta} = g_{\alpha\beta}|_{\gamma} dx^\gamma + g_{\alpha\beta}|_{\gamma} Dl^\gamma
\end{equation}

\begin{equation}
  Dl^\gamma = dl^\gamma + \Gamma^\gamma_{\alpha \beta} dx^\beta + A^\gamma_{\alpha \beta} Dl^\beta,
\end{equation}

where $D$ denotes the absolute differential which corresponds to the change of the lineelement from $(x, \dot{x})$ to $(x + dx, \dot{x} + d\dot{x})$ and $"\circ"$ means the contraction by $l$. The connection coefficients $\Gamma^\alpha$ and $A$ are determined under conditions

\begin{equation}
  \Gamma^\alpha_{\alpha \gamma} = \Gamma^\alpha_{\gamma \alpha}
\end{equation}

\begin{equation}
  A_{\alpha \gamma} = A_{\gamma \alpha}.
\end{equation}

AMS Subject Classification (1980): Primary 53B40.
From (1.1) and (1.5) \( \Gamma_{\alpha\beta}^\gamma \) may be determined in the unique way and similarly (1.2) and (1.6) determine \( A_{\alpha\beta\gamma} \). The connection coefficients obtained in this way are generalisations of the Cartan connections in the case of a non recurrent Finsler space (when \( \lambda_\gamma = 0 \) and \( \mu_\gamma = 0 \)).

Using the notation \( \{ T_{\gamma\alpha\beta} \} + \{ \gamma\alpha\beta \} = T_{\gamma\alpha\beta} + T_{\alpha\beta\gamma} - T_{\beta\gamma\alpha} \) we have [3]

\[
2 \Gamma_{\alpha\beta}^\gamma = \{ \partial_\gamma \, g_{\alpha\beta} - F \partial_\delta \, g_{\alpha\beta} \Gamma_{\delta}^{\gamma} - \lambda_\gamma \, g_{\alpha\beta} \} + \{ \gamma\alpha\beta \}
\]

(1.7)

\[
2 \Gamma^*_{\alpha\beta\gamma} = 2 \gamma_{\alpha\beta\gamma} l^\gamma - F \partial_\delta \, g_{\beta\gamma} \Gamma_{\delta}^{\gamma} - (\lambda_\gamma l_\beta + \lambda_\beta l_\gamma - \lambda_\gamma l_\beta - \lambda_\beta l_\gamma)
\]

(1.8)

where \( \gamma_{\alpha\beta\gamma} \) is the Christoffel symbol. Further we obtain

\[
2 A_{\alpha\beta\gamma} = \{ F \partial_\alpha \, g_{\beta\gamma} - F \partial_\delta \, g_{\beta\gamma} \, A_{\delta}^{\alpha} - \mu_\alpha \, g_{\alpha\beta} \} + \{ \alpha\beta\gamma \}
\]

(1.10)

\[
2 A^*_{\alpha\beta\gamma} = - F \partial_\delta \, g_{\beta\gamma} \, A_{\delta}^{\alpha} - (\mu_\alpha g_{\beta\gamma} + \mu_\gamma l_\beta - \mu_\beta l_\gamma)
\]

(1.11)

\[
2 A_{\alpha\beta\gamma} = -(2\mu_\alpha l_\beta - \mu_\beta)
\]

(1.12)

We shall suppose that in \( \mathcal{F}_n \) all vector and tensor fields are homogeneous of degree zero in \( \hat{x} \).

**Lemma 1.1.** If in \( \mathcal{F}_n \) \( \xi^\alpha_{|\beta} \) and \( \xi^\alpha_{|\beta} \) are defined by

\[
\xi^\alpha_{|\beta} = \partial_\beta \xi^\alpha - F \partial_\delta \xi^\alpha \Gamma_{\delta}^{\gamma}_{\beta\gamma} + \Gamma^{*\gamma}_{\delta\beta} \xi^\delta
\]

(1.13)

\[
\xi^\alpha_{|\beta} = F \partial_\delta \xi^\alpha (\delta^\delta_{\beta} - A_{\delta}^{\alpha}) + A_{\delta}^{\alpha} \xi^\delta,
\]

(1.14)

then

\[
\xi_{\alpha|\beta} = \partial_\beta \xi_{\alpha} - F \partial_\delta \xi_{\alpha} \Gamma_{\delta}^{\gamma}_{\beta\gamma} - \Gamma^{*\gamma}_{\delta\beta} \xi_{\delta}
\]

(1.15)

\[
\xi_{\alpha|\beta} = F \partial_\delta \xi_{\alpha} (\delta^\delta_{\beta} - A_{\delta}^{\alpha}) - A_{\delta}^{\alpha} \xi_{\delta}
\]

(1.16)

**Proof.** From \( \xi_{\alpha|\beta} = (g_{\alpha\delta} \xi^\delta_{|\beta}) = g_{\alpha|\beta} \xi^\delta + g_{\alpha\delta} \xi^\delta_{|\beta} \) by using (1.13) (1.1) and

\[
g_{\alpha\delta} \partial_\beta \xi^\delta = \partial_\beta \xi_{\alpha} - \xi^\delta \partial_\beta g_{\delta\alpha}
\]

(1.17)

we obtain (1.15). From

\[
\xi_{\alpha|\beta} = (g_{\alpha\delta} \xi^\delta_{|\beta}) = g_{\alpha\delta} \xi^\delta + g_{\alpha\delta} \xi^\delta_{|\beta}
\]

(1.17)

by using (1.16) (1.2) and (1.17) we have (1.16).
Using the notations of (1.13)–(1.16) we have
\[ D\xi^\alpha = \xi^\alpha_{\beta \gamma} dx^\beta + \xi^\alpha_{\beta} Dl^\beta, \quad D\xi_\alpha = \xi_{\alpha \beta \gamma} dx^\beta + \xi_{\alpha \beta} Dl^\beta. \]

**Lemma 1.2.** In \( \overline{\mathbb{R}}^n \) vector \( dx \) is normal to \( \lambda \) iff \( \mu + 2l \) is normal to \( Dl \) i.e.
\[ \lambda_\gamma dx^\gamma = 0 \Leftrightarrow (\mu_\gamma + 2l_\gamma) Dl^\gamma = 0. \]

**Proof.** From \( g_{\alpha \beta} l^\alpha l^\beta = 1 \) we get \( Dg_{\alpha \beta} l^\alpha l^\beta + g_{\alpha \beta} Dl^\beta = 0. \)
Using (1.3), (1.1) and (1.2) we have
\[ \lambda_\gamma dx^\gamma = 0 \Leftrightarrow (\mu_\gamma + 2l_\gamma) Dl^\gamma = 0. \]
from which the statement follows.

An obvious consequence of (1.18) is:

**Lemma 1.3.** If the vector \( l \) is parallelly transported from \((x, \dot{x})\) to \((x + dx, \dot{x} + d\dot{x})\) i.e. \( Dl^\gamma = 0 \) then \( \lambda_\gamma dx^\gamma = 0 \), which means that \( dx \) is normal to \( \lambda \).

For any vector field \( \xi^\alpha(x, \dot{x}) \) we have
\[ D\xi^\alpha = d\xi^\alpha + w^\alpha_\beta(d)\xi^\beta \]
where
\[ w^\alpha_\beta(d) = \Gamma^\alpha_\beta_\gamma dx^\gamma + A^\alpha_\beta_\gamma Dl^\gamma \]
From (1.4) we obtain
\[ Dl^\beta I^\gamma_\delta = dl^\gamma + \Gamma^\gamma_\beta_\delta dx^\beta \]
where \( I^\gamma_\delta = \delta^\gamma_\delta - A^\gamma_\delta \).

Let us suppose that \([I^\gamma_\delta]\) is a regular matrix whose inverse is \([J^\theta_\gamma]\)
\[ I^\gamma_\delta J^\theta_\gamma = \delta^\theta_\gamma \]

From (1.21) it follows \( Dl^\theta = (dl^\gamma + \Gamma^\gamma_\delta_\nu dx^\nu)J^\theta_\gamma \).
Further from \( l^\nu = F^{-1} \dot{x}^\nu \) and
\[ dl^\nu = (\partial_\nu F^{-1} dx^\gamma - F^{-1} l_\gamma dx^\gamma) \dot{x}^\nu + F^{-1} \dot{\dot{x}}^\nu \]
we have
\[ Dl^\theta = J^\theta_\chi[(\Gamma^\gamma_\theta_\chi - F^{-1} \partial_\chi F) dx^\gamma + (\delta^\gamma_\chi - l_\gamma l^\chi) \dot{x}^\gamma]. \]

**2. Connection coefficients \( \Gamma \) and \( C. \)** \( w^\alpha_\beta(d) \) appearing in (1.19) and (1.20) may be written in the form
\[ w^\alpha_\beta(d) = \Gamma^\alpha_\beta_\gamma dx^\gamma + C^\alpha_\beta_\gamma d\dot{x}^\gamma. \]
The connection coefficients $\Gamma^\gamma_{\beta\gamma}$ and $A$ from (1.20) are uniquely determined under conditions (1.1), (1.2), (1.5) and (1.6). They are given by (1.7)--(1.12). We are going to obtain relations between $F$, $C$ and $\Gamma^\gamma_{\beta\gamma}$ and $A$. For that reason we shall equate the right hand side of (1.20) and (2.1) and use the relations (1.18), (1.23) and obtain

$$
\Gamma^\alpha_{\beta\gamma} dx^\gamma + C^\alpha_{\beta\gamma} d\hat{x}^\gamma = \Gamma^{*\alpha}_{\beta\gamma} dx^\gamma + A^\alpha_{\beta\gamma} Dl^\gamma + \theta^\alpha_{\beta\gamma} [\lambda_\gamma dx^\gamma + (\mu_\gamma + 2l^\gamma) Dl^\gamma]
$$

or

$$
\Gamma^\alpha_{\beta\gamma} dx^\gamma + C^\alpha_{\beta\gamma} d\hat{x}^\gamma = (\Gamma^{*\alpha}_{\beta\gamma} + \theta^\alpha_{\beta\gamma}) dx^\gamma
$$

(2.2)

$$
[A^\alpha_{\beta\gamma} + \theta^\alpha_{\beta\gamma}] J^\alpha_{\beta\gamma} = (\Gamma^{*\alpha}_{\beta\gamma} + \theta^\alpha_{\beta\gamma}) dx^\gamma
$$

where $\theta^\alpha_{\beta\gamma} = \theta^\alpha_{\beta\gamma}(x, \hat{x})$ is any tensor homogeneous of degree zero in $\hat{x}$. By equating the coefficients beside $dx^\gamma$ and $d\hat{x}^\gamma$ we obtain

$$
\Gamma^\alpha_{\beta\gamma} = \gamma^\alpha_{\beta\gamma} + \theta^\alpha_{\beta\gamma} + [A^\alpha_{\beta\gamma} + \theta^\alpha_{\beta\gamma}] J^\alpha_{\beta\gamma} (\Gamma^{\alpha\chi}_{\gamma} - F^{-1} l^\chi \partial_\gamma F),
$$

(2.3)

$$
C^\alpha_{\beta\gamma} = [A^\alpha_{\beta\gamma} + \theta^\alpha_{\beta\gamma}] J^\alpha_{\beta\gamma} F^{-1} (\delta^\chi_\gamma - l^\chi)
$$

(2.4)

**Lemma 2.1.** The relation

(2.5)

$$
C^\alpha_{\beta\gamma} \hat{x}^\gamma = FC^\alpha_{\beta\gamma} = 0
$$

is valid for any $\theta^\alpha_{\beta\gamma}$.

The proof is obvious from (2.4).

For $\theta^\alpha_{\beta\gamma} = 0$, (2.3) and (2.4) become [4]

$$
\Gamma^\alpha_{\beta\gamma} = \gamma^\alpha_{\beta\gamma} + A^\alpha_{\beta\gamma} J^\alpha_{\beta\gamma} (\Gamma^{\alpha\chi}_{\gamma} - F^{-1} l^\chi \partial_\gamma F)
$$

(2.6)

$$
C^\alpha_{\beta\gamma} = A^\alpha_{\beta\gamma} J^\alpha_{\beta\gamma} F^{-1} (\delta^\chi_\gamma - l^\chi)
$$

(2.7)

Formulae (2.6) and (2.7) are not practical for calculation because they contain the term $J^\alpha_{\beta\gamma}$, for which all we know is the relation (1.22).

From (1.21) we obtain

$$
d\hat{x}^\gamma = F(\delta^\gamma_\theta - A_\gamma^\alpha \theta^\alpha_{\beta\gamma}) dl^\theta - F R^\alpha_{\beta\gamma}^\delta dx^\delta - \hat{x}^\gamma F dF^{-1}
$$

(2.8)

Substituting (2.8) into (2.2) we have

$$
\Gamma^\alpha_{\beta\gamma} - FC^\alpha_{\beta\delta} \Gamma^{\alpha\delta}_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} + \theta^\alpha_{\beta\gamma}
$$

(2.9)

$$
FC^\alpha_{\beta\delta} (\delta^\delta_\gamma - A_\gamma^\alpha \theta^\alpha_{\beta\gamma}) = A^\alpha_{\beta\gamma} + \theta^\alpha_{\beta\gamma}(\mu_\gamma + 2l_\gamma)
$$

(2.10)

In the case of non recurrent Finsler space where $\lambda_\gamma = 0$, $\mu_\gamma = 0$, $A_\gamma^\delta = 0$ the equations (2.9) and (2.10) have the form

$$
\Gamma^\alpha_{\beta\gamma} - FC^\alpha_{\beta\delta} \Gamma^{\alpha\delta}_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma}
$$

(2.11)
\[ FC^\alpha_{\beta\gamma} = A^\alpha_{\beta\gamma} + 2\theta^\alpha_{\beta\gamma} \]

For \( \theta^\alpha_{\beta\gamma} = 0 \) (2.12) takes the well known form \( FC^\alpha_{\beta\gamma} = A^\alpha_{\beta\gamma} \). In the further calculation we shall use the formulae [4]

\[ F_{\gamma} = \partial_{\gamma} F - F \partial_{\gamma} F \Gamma^\delta_{\gamma} = 2^{-1} F \lambda_{\gamma}. \]

3. **Parallel displacement of vector along the cycle of linelements.**

Let us consider the cycle of linelements as they are presented on the picture

\[ P_1(x + \delta x, x + \delta \delta x) \]
\[ P_3 = P_1 + dP_1 = (x + \delta x + d x, x + \delta x + d \delta x) \]
\[ P_2 = P_1 + \delta P_1 = (x + d x + \delta x, x + \delta x + \delta \delta x) \]
\[ P_1(x + dx, x + d\delta x) \]

Let us fix the point \( P \) with the local coordinates \( x^\alpha \) in \( T_n \). By \( T_n(P) \) we shall denote the set of all \( \tilde{x} \) in \( P \) which form a tangent space. In \( T_n(P) \) we can construct a basis which contains the tangent vectors \( r_\alpha (\alpha = 1, 2, \ldots, n) \) on the coordinate curves \( x^\beta = C^\beta, \beta = 1, 2, \ldots, n \). Let us consider two infinitesimal vectors \( PP_1 \) and \( PP_2 \) which respectively have the form \( PP_1 = dx^\alpha r_\alpha \), \( PP_2 = dx^\alpha \delta r_\alpha \). If the vector \( PP_1 \) is parallelly transported along \( PP_2 \) we get the point \( P_3 \) and if \( PP_2 \) is parallelly moved along \( PP_1 \) we get \( P_2 \). In this case the linelement are not parallel, only the basic vectors are. The coordinates of the point \( P_3 \) are \( x^\alpha + dx^\alpha + \delta dx^\alpha \), \( \delta dx^\alpha = -w_3^\alpha(\delta)dx^\beta \). The coordinates of the point \( P_2 \) are \( x^\alpha + \delta x^\alpha + dx^\alpha + d\delta x^\alpha \) where \( d\delta x^\alpha = -w_2^\alpha(\delta)dx^\beta \). In the general case \( P_3 \) and \( P_2 \) are not the same points and the vector \( P_3 P_2 \) is the torsion vector in \( T_n \). It has the coordinates

\[ \Omega^\alpha = d\delta x^\alpha - \delta dx^\alpha = w_2^\alpha(\delta)dx^\beta - w_3^\alpha(\delta)dx^\beta \]

In \( T_n \) with the connection coefficients \( \Gamma^\alpha_{\beta\gamma} \) and \( A \) we obtain

\[ \Omega^\alpha = A^\alpha_{\beta\gamma}(dx^\beta \Delta \gamma - \delta x^\beta D\gamma). \]

If \( D\gamma = 0 \) and \( \Delta \gamma = 0 \), then \( \Omega^\alpha = 0 \) and the points \( P_3 \) and \( P_2 \) have the same coordinates. In that case we have an infinitesimal parallelogram \( PP_1 P_2 P_3 \).

Let us consider how the basic vectors change if they are parallelly transported along \( PP_1 P_3 \) and \( PP_2 P_3 \).

By the parallel transportation of \( r_\alpha \) from \( P(x, \tilde{x}) \) to \( P_1(x + dx, \tilde{x} + d\tilde{x}) \) we obtain in \( P_1 r_\alpha + dr_\alpha \), where \( Dr_\alpha = dr_\alpha - w_\alpha^{\beta}(d)r_\beta \) = 0.
By the parallel transportation of \( r_\alpha \) from \( P(x, \dot{x}) \) to \( P_2(x + \delta x, \dot{x} + \delta \dot{x}) \) in \( P_2 \) we get \( r_\alpha + \delta r_\alpha \), where \( \Delta r_\alpha = \delta r_\alpha - w_\alpha^\beta(d)r_\beta = 0. \)

If the vector \( r_\alpha + \delta r_\alpha \) at \( P_1 \) is parallelly transported to \( P_2(x + \delta x + \dot{d}x, \dot{x} + \delta \dot{x} + \delta \dot{d}) \) at \( P_3 \) we have the vector \( r_\alpha + \delta r_\alpha + \delta(r_\alpha + \delta r_\alpha) \), where

\[
\delta \delta r_\alpha = \delta w_\alpha^\beta(d)r_\beta + w_\alpha^\delta(d)w_\delta^\beta(\delta)r_\beta.
\]

If the vector \( r_\alpha + \delta r_\alpha \) at \( P_2 \) is parallelly transported to \( P_3(x + \delta x + \dot{d}x, \dot{x} + \delta \dot{x} + \delta \dot{d}) \) at \( P_3 \) we get the vector \( r_\alpha + \delta r_\alpha + d(r_\alpha + \delta r_\alpha) \) where

\[
d \delta r_\alpha = dw_\alpha^\beta(\delta)r_\beta + w_\alpha^\beta(\delta)w_\beta^\delta(d)r_\beta.
\]

If the vector \( r_\alpha + \delta r_\alpha + d r_\alpha + d \delta r_\alpha \) at \( P_2 \) is parallelly transported to \( P_3 \) we obtain in \( P_3 \) the vector \( r_\alpha + \delta r_\alpha + d r_\alpha + d \delta r_\alpha + \nabla r_\alpha \) where \( \nabla r_\alpha \) describes the change of \( r_\alpha \) along \( P_3 \) and has the form

\[
\nabla r_\alpha = \Gamma_\alpha^\beta \gamma^\gamma r_\beta(\delta d - \delta d) x^\gamma + C_\alpha^\beta \gamma^\gamma r_\beta(\delta d - \delta d) \dot{x}^\gamma.
\]

The difference between vectors which are obtained by parallel transportation of \( r_\alpha \) along \( PP_2PP_3 \) and \( P_2P_3 \) is denoted by \( \overline{Dr}_\alpha \). Then we have

\[
\overline{Dr}_\alpha = - (\delta d - \delta d) r_\alpha + \nabla r_\alpha = - (\delta d - \delta d) r_\alpha + \Gamma_\alpha^\beta \gamma^\gamma r_\beta(\delta d - \delta d) x^\gamma + C_\alpha^\beta \gamma^\gamma r_\beta(\delta d - \delta d) \dot{x}^\gamma.
\]

The vector \( \overline{Dr}_\alpha \) can be expressed by the curvature tensors. We have \( Dr_\alpha = \delta r_\alpha - w_\alpha^\beta(d)r_\beta \) and

\[
\Delta D r_\alpha = \delta (Dr_\alpha) - w_\alpha^\delta(\delta) Dr_\delta = \delta \delta r_\alpha - \delta w_\alpha^\beta(d)r_\beta - w_\alpha^\delta(d) r_\beta - w_\alpha^\delta(\delta)[dr_\delta - w_\delta^\beta(d)r_\beta].
\]

From the above equation we get

\[
(\Delta D - DD)r_\alpha = (\delta d - \delta d)r_\alpha - \Omega_\alpha^\beta r_\beta,
\]

where

\[
\begin{align*}
w_\alpha^\beta &= \left[ w_\alpha^\delta w_\delta^\beta \right] - (w_\alpha^\beta)^\gamma, \\
[w_\alpha^\delta w_\delta^\beta] &= w_\alpha^\delta(d) w_\delta^\beta(\delta) - w_\alpha^\delta(\delta) w_\delta^\beta(d), \\
(w_\alpha^\beta)^\gamma &= \delta w_\alpha^\beta(d) - d w_\alpha^\beta(\delta).
\end{align*}
\]

After some calculation we obtain

\[
\Omega_\alpha^\beta = A_\alpha^\beta + B_\alpha^\beta,
\]

where \([5]\)

\[
A_\alpha^\beta = 2^{-1} K_\alpha^\beta \gamma^\delta [dx^\beta \delta x^\gamma] + (P_\alpha^\beta \gamma^\delta - A_\alpha^\beta \gamma^\delta \partial_\delta \Gamma_\delta^\gamma) + 2^{-1} S_\alpha^\beta \gamma^\delta [DF^\gamma \Delta^\delta]
\]
\(B_\alpha^\beta = A_\alpha^\beta \gamma (\delta D - d\Delta) l^\gamma + \Gamma_{\alpha\gamma}^\beta (\delta d - d\delta) x^\gamma\)

\[2^{-1} K_{\alpha^\gamma \beta} \gamma = \partial_\delta^\gamma \Gamma_{\alpha\gamma}^\beta - \partial_\beta^\gamma \Gamma_{\alpha^\gamma}^\beta + \Gamma_{\alpha^\gamma}^\beta \Gamma_{\alpha\gamma}^\beta \gamma.
\]

\(P^\gamma_{\alpha^\gamma} \beta \gamma = F \partial_\delta^\gamma \Gamma_{\alpha^\gamma}^\beta (\delta \delta - A_0^\beta) - A_\alpha^\beta \gamma - A_\alpha^\beta \gamma \partial_\delta^\gamma \partial_\beta^\gamma + \partial_\delta^\gamma A_\alpha^\beta \gamma + A_\alpha^\beta \gamma A_0^\beta \gamma + A_\alpha^\beta A_0^\beta \gamma + A_0^\beta A_\alpha^\beta \gamma.
\]

On the other hand from (1.4) and (2.8) using the homogeneity of \(\Gamma_{\gamma^\gamma}^\gamma = F \Gamma_{\gamma^\gamma}^\gamma\) (first degree) and \(A_0^\gamma (zero degree) we obtain
\[(\delta x - A_0^\gamma) (\delta D - d\Delta) l^\delta = B^\gamma + \overline{B}^\gamma
\]

where
\[
\overline{B}^\gamma = F^{-1} \left( \partial_\delta^\gamma \Gamma_{\gamma^\gamma}^\gamma - \partial_\delta^\gamma \Gamma_{\gamma^\gamma}^\gamma \Gamma_{\gamma^\gamma}^\gamma \right) + \left( \partial_\delta^\gamma \Gamma_{\gamma^\gamma}^\gamma + \partial_\delta^\gamma \Gamma_{\gamma^\gamma}^\gamma A_0^\gamma - \partial_\delta^\gamma \partial_\delta^\gamma + \partial_\delta^\gamma \partial_\delta^\gamma \partial_\delta^\gamma \right) [d\delta \gamma \delta d \delta] + \Gamma_{\gamma^\gamma}^\gamma \partial_\delta^\gamma \partial_\gamma^\gamma A_0^\gamma
\]

\[(\delta D - d\Delta) \partial \gamma^\gamma + \partial \gamma^\gamma (\delta D - d\Delta) F^{-1} + F^{-1} \Gamma_{\gamma^\gamma}^\gamma (\delta D - d\Delta) x^\gamma
\]

It is known that \(\dot{\gamma}^\gamma \gamma\), so from the above equation and (3.4) we obtain
\[2^{-1} K_{\alpha^\gamma \beta} \gamma = F^{-1} \left( \partial_\delta^\gamma \Gamma_{\gamma^\gamma}^\gamma - \partial_\delta^\gamma \Gamma_{\gamma^\gamma}^\gamma \Gamma_{\gamma^\gamma}^\gamma \right)
\]

Substituting (2.10) into (3.5) we get
\[(3.10) \quad B_\alpha^\beta = B_\alpha^\beta (1) + B_\alpha^\beta (2)
\]

where according to (3.7) we have
\[B_\alpha^\beta (1) = \Gamma_{\alpha\gamma}^\beta (\delta d - d\delta) x^\gamma + F C_\alpha^\beta \gamma B_\gamma^\gamma - \theta_\gamma^\gamma \beta (\mu_\delta + 2\mu_\delta) (\delta D - d\Delta) l^\delta,
\]

\[B_\alpha^\beta (2) = F C_\alpha^\beta \gamma \overline{B}^\gamma
\]

From (1.18) we get
\[\lambda_\gamma (\delta d - d\delta) x^\gamma + (\mu_\gamma + 2\mu_\gamma) (\delta D - d\Delta) l^\gamma + (\delta \lambda_\gamma d x^\gamma - d \lambda_\gamma \delta x^\gamma + \delta (\mu_\gamma + 2\mu_\gamma) D l^\gamma - d (\mu_\gamma + 2\mu_\gamma) \Delta l^\gamma = 0
\]

and using (3.18) and (2.5) we have
\[B_\alpha^\beta (1) = (\Gamma_{\alpha\gamma}^\beta + C_\alpha^\beta \gamma \Gamma_{\alpha\gamma}^\gamma + \theta_\gamma^\gamma \lambda_\gamma) (\delta d - d\delta) x^\gamma
\]

\[C_\alpha^\beta \gamma (\delta d - d\delta) \dot{x}^\gamma + B_\alpha^\beta (1) \gamma
\]

\[B_\alpha^\beta (1) = \theta_\gamma^\gamma [\delta \lambda_\gamma d x^\gamma - d \lambda_\gamma \delta x^\gamma
\]

\[\delta (\mu_\gamma + 2\mu_\gamma) D l^\gamma - d (\mu_\gamma + 2\mu_\gamma) \Delta l^\gamma].
\]
Substituting $\Gamma^\beta_{\alpha\gamma}$ from (2.9) into (3.13) we have
\[ B^\beta_{\alpha} (1) = \Gamma^\beta_{\alpha\gamma} (\delta d - \delta d) x^\gamma + C^\beta_{\alpha\gamma} (\delta d - \delta d) x^\gamma + B^\beta_{\alpha} (1)'. \]

Using (3.9) and the relation
\[ \delta^i, \Gamma^\beta_{\alpha\gamma} (\delta^i - A^i, \delta) - \partial^i A^i, \delta + \delta^i, A^i, \delta \Gamma^\beta_{\alpha\gamma} = \]
\[ P^\beta_{\alpha\gamma} (\delta^i - A^i, \delta) - \partial^i A^i, \delta + \delta^i, A^i, \delta \Gamma^\beta_{\alpha\gamma} = \]
\[ 2^{-1} A^i, \delta \lambda \gamma, \]
the formula (3.11) has the form
\[ B^\beta_{\alpha} (2) = FC^\beta_{\alpha\gamma} [2^{-1} K^\beta_{\alpha\gamma} - dy^\gamma \Delta t^\delta] + \]
\[ (P^\beta_{\alpha\gamma} - A^i, \delta) \delta^i, A^i, \lambda \gamma + 2^{-1} A^i, \delta \lambda \gamma, ] [d\gamma \Delta t^\delta] + \]
\[ 2^{-1} (\partial^i, A^i, \lambda \gamma (\partial^i - A^i, \partial^i) [dy^\gamma \Delta t^\delta]). \]

Theorem 3.1. In the recurrent Finsler space $F_n$ and the curvature tensors are connected by:
\[ (\Delta D - D \Delta) r_{\alpha} = -\overline{D} r_{\alpha} - r_{\beta} \{ 2^{-1} [K^\beta_{\alpha\gamma} - FC^\beta_{\alpha\gamma} K^\delta_{\alpha\gamma \delta}] [dy^\gamma \Delta t^\delta] + \]
\[ \{ P^\beta_{\alpha\gamma} - A^i, \delta \delta^i, A^i, \lambda \gamma + FC^\beta_{\alpha\gamma} (P^\beta_{\alpha\gamma} - A^i, \delta) \delta^i, A^i, \lambda \gamma + 2^{-1} A^i, \delta \lambda \gamma, ] [d\gamma \Delta t^\delta] + \]
\[ 2^{-1} [S^\beta_{\alpha\gamma} + FC^\beta_{\alpha\gamma} (\partial^i - A^i, \partial^i) [dy^\gamma \Delta t^\delta] - r_{\beta} B^\beta_{\alpha} (1)^{'}]. \]

Proof. Substituting (3.16), (3.13), (3.14) into (3.10), further (3.10) and (3.4) into (3.3), (3.4) into (3.2) by using (3.1) we obtain (3.17).

In the non recurrent Finsler space (where $\lambda, \gamma = 0$ and $\mu, \gamma = 0$ we have
\[ B^\beta_{\alpha} (1) = 2\theta^\beta_{\alpha} (\delta l_{\gamma} Dl^\gamma - dl_{\gamma} \Delta l^\gamma) \]

If we have not only $\lambda, \gamma = 0$, $\mu, \gamma = 0$ but the condition $\theta^\beta_{\alpha} = 0$, then the connection coefficients $A^\beta_{\alpha\gamma}$ and $\Gamma^\beta_{\alpha\gamma}$ are the Cartan's connection coefficients and $A^\beta_{\alpha\gamma} = FC^\beta_{\alpha\gamma}$. In this case from (1.11), (1.12) it follows $\alpha, \lambda, \gamma = 0$ the left hand side of (3.15) reduces to the $\partial^i, A^i, \lambda \gamma$ and (3.17) has the form
\[ (\Delta D - D \Delta) r_{\alpha} = \]
\[ -\overline{D} r_{\alpha} - r_{\beta} \{ 2^{-1} R^\beta_{\alpha\gamma} [dy^\gamma \Delta t^\delta] + P^\beta_{\alpha\gamma} \gamma \delta [dy^\gamma \Delta t^\delta] + 2^{-1} S^\beta_{\alpha\gamma} \gamma \delta [Dl^\gamma \Delta t^\delta]). \]

When the vector $r_{\alpha}$ is parallely transported along $PP_{1} P_{2}$ and $PP_{2} P_{3}$ then $D r_{\alpha} = 0$, $\Delta r_{\alpha} = 0$ and in this case from (3.18) we have
\[ -\overline{D} r_{\alpha} = -r_{\beta} \{ 2^{-1} R^\beta_{\alpha\gamma} \gamma \delta [dy^\gamma \Delta t^\delta] + P^\beta_{\alpha\gamma} \gamma \delta [dy^\gamma \Delta t^\delta] + 2^{-1} S^\beta_{\alpha\gamma} \gamma \delta [Dl^\gamma \Delta t^\delta]). \]

In the case of a recurrent Finsler space $\overline{F_n}$ when $D r_{\alpha} = 0$ and $\Delta r_{\alpha} = 0$ from (3.17) $\overline{D} r_{\alpha}$ has more complicated form.
4. **Special cases** Case 1. Let us consider the case when in $\mathfrak{F}_n$, $dx^\gamma = 0$ and $\delta x^\gamma = 0$ i.e. when the linelements $P_1$, $P_2$ and $P_3$ have the common center $x$.

Then we have

$$P(x, \dot{x}), \quad P_1(x, \dot{x} + d\dot{x}), \quad P_2(x, \dot{x} + \delta \dot{x})$$

$$P_3 = P_1 \quad \text{and} \quad P_5 = P_2 + dP_2 = (x, \dot{x} + d\dot{x} + \delta\dot{x}).$$

In this case we have

$$Dr_\alpha = dr_\alpha - A_\alpha^\beta r_\beta Dl^\gamma, \quad \Delta r_\alpha = \delta r_\alpha - A_\alpha^\beta r_\beta \Delta l^\gamma$$

and

$$(\Delta - D\Delta) r_\alpha = (\delta d - d\delta) r_\alpha - 2^{-1} r_\beta [F \hat{\partial}_\alpha A_\alpha^\beta \gamma \delta \beta - A_\alpha^\beta \gamma \delta \beta] +$$

$$A_\alpha^\beta \gamma [\delta [Dl^\gamma \Delta l^\delta] - A_\alpha^\beta \gamma r_\beta (\delta D - d\Delta) l^\gamma].$$

Substituting $A_\alpha^\beta \gamma$ from (2.10) and using (3.12) where $(\delta d - d\delta) x^\gamma = 0$ we have

$$- A_\alpha^\beta \chi r_\beta (\delta D - d\Delta) l^\chi = - F C_\alpha^\beta r_\beta (\delta^\chi - A_\alpha^\beta \gamma) (\delta D - d\Delta) l^\chi$$

$$- \theta_\alpha^\beta r_\beta [\delta (\mu + 2l_\gamma) Dl^\gamma - d(\mu + 2l_\gamma) \Delta l^\gamma].$$

As in this case

$$(\delta^\chi - A_\alpha^\beta \gamma) Dl^\chi = dl^\chi, \quad (\delta^\chi - A_\alpha^\beta \gamma) \Delta l^\chi = \delta l^\chi$$

using the homogeneity condition we obtain

$$(\delta^\chi - A_\alpha^\beta \gamma)(\delta D - d\Delta) l^\chi = F^{-1} (\delta d - d\delta) \dot{x^i} + \dot{x^i}(\delta d - d\delta) F^{-1} +$$

$$F \hat{\partial}_\chi A_\alpha^\beta \gamma (\delta^\chi - A_\alpha^\beta \gamma)(Dl^\gamma \Delta l^\delta - \Delta l^\gamma \Delta l^\delta].$$

Substituting (4.3) into (4.2) and then (4.2) into (4.1) we get

$$(\Delta D - D\Delta) r_\alpha = - \overline{Dr}_\alpha - r_\beta [2^{-1} S_\alpha^\beta \gamma \delta + F C_\alpha^\beta \chi \hat{\partial}_\alpha \delta \beta \gamma \delta (\delta^\chi - A_\alpha^\beta \gamma)][Dl^\gamma \Delta l^\delta] -$$

$$\theta_\alpha^\beta r_\beta [\delta (\mu + 2l_\gamma) Dl^\gamma - d(\mu + 2l_\gamma) \Delta l^\gamma].$$

where from (3.1) in this case $\overline{Dr}_\alpha$ has the form

$$\overline{Dr}_\alpha = -(\delta d - d\delta) r_\alpha + C_\alpha^\beta \gamma r_\beta (\delta d - d\delta) \dot{x}^\gamma$$

In the non-recurrent Finsler space $F_n$, where we take $\theta_\alpha^\beta = 0$, $\mu_\gamma = 0$ we have

$$A_\alpha^\beta \gamma = F C_\alpha^\beta \gamma \Rightarrow A_0^\beta \gamma = 0$$

where

$$(\Delta D - D\Delta) r_\alpha = - \overline{Dr}_\alpha - 2^{-1} r_\beta S_\alpha^\beta \gamma [Dl^\gamma \Delta l^\delta].$$

In the case when $Dr_\alpha = 0$, $\Delta r_\alpha = 0$ (4.4) gives

$$\overline{Dr}_\alpha = - 2^{-1} r_\beta S_\alpha^\beta \gamma [Dl^\gamma \Delta l^\delta].$$
Case 2. Let us consider the line elements
\[ P(x, \dot{x}) \]
\[ P_1(x + dx, \dot{x} + \delta \dot{x}) \quad \text{with} \quad Dl = 0 \]
\[ P_2(x, \dot{x} + \delta \dot{x}) \quad \text{with} \quad \delta x = 0 \]
\[ P_3 = P_1 + \delta P_1 = (x + dx, \dot{x} + \delta \dot{x} + \delta d\dot{x}), \quad (\delta x = 0), \]
\[ P_3^\gamma = P_2 + dP_2 = (x + dx, \dot{x} + \delta \dot{x} + d\delta \dot{x}). \]

From \( Dl^\delta = 0 \) we have
\[ d\dot{x}^\delta = -F \dot{x}^\delta dF^{-1} - \Gamma^\gamma_\delta dx^\gamma. \tag{4.5} \]

From \( \delta x^\delta = 0 \) we get
\[ (\delta^\delta - A^\delta_0) \Delta l^\delta = \delta l^\delta = F^{-1} \dot{x}^\delta + \dot{x}^\delta F^{-1} \Rightarrow \]
\[ \delta \dot{x}^\delta = (\delta^\delta - A^\delta_0) \Delta l^\delta - F \dot{x}^\delta F^{-1}. \tag{4.6} \]

In this case we have
\[ (\delta^\delta - A^\delta_0) \Delta l^\delta = \delta l^\delta = F^{-1} \dot{x}^\delta + \dot{x}^\delta F^{-1} \Rightarrow \]
\[ \delta \dot{x}^\delta = (\delta^\delta - A^\delta_0) \Delta l^\delta - F \dot{x}^\delta F^{-1}. \tag{4.7} \]

a) \( Dr_\alpha = dr_\alpha - \Gamma^\alpha_\gamma r_\beta \delta x^\gamma \)

b) \( \Delta r_\alpha = \delta r_\alpha - A^\delta_\gamma r_\beta \Delta l^\gamma \)

From \( \delta x = 0 \Rightarrow d\delta x = 0 \) and \( Dr_\alpha \) has the form
\[ -Dr_\alpha = -(\delta d - d\delta) r_\alpha + \Gamma^\alpha_\gamma r_\beta \delta x^\gamma - C^\alpha_\gamma r_\beta (\delta d - d\delta) \dot{x}^\gamma \tag{4.8} \]

From \( (4.7) \) we obtain
\[ (\Delta D - D \Delta) r_\alpha = r_\beta [F \dot{\delta} \Gamma^\alpha_\gamma (\delta^\delta - A^\delta_0) - \partial_\gamma A^\delta_\beta + \partial_\beta A^\delta_\gamma \Gamma^\gamma_\delta \]
\[ -A^\delta_\gamma \Gamma^\gamma_\delta + A^\delta_\beta \Gamma^\gamma_\delta \] \( dx^\gamma \Delta l^\delta + \]
\[ (\delta d - d\delta) r_\alpha - \Gamma^\alpha_\gamma r_\beta d\delta \dot{x}^\gamma + A^\alpha_\gamma r_\beta d\Delta l^\gamma. \tag{4.9} \]

From (2.10) using (4.6) and \( C^\alpha_\gamma \dot{x}^\gamma = 0 \) we get
\[ A^\alpha_\beta r_\beta d\Delta l^\gamma = [FC^\alpha_\beta r_\beta (\delta^\delta - A^\delta_0) - \theta^\beta_\alpha (\mu_\gamma + 2l_\gamma)] d \Delta l^\gamma. \]

From (3.12) in case 2 it follows
\[ B = (\mu_\gamma + 2l_\gamma) d \Delta l^\gamma = \lambda_\gamma \delta dx^\gamma + d\lambda_\gamma dx^\gamma - d(\mu_\gamma + 2l_\gamma) \Delta l^\gamma. \]

From Lemma 1.3 it follows that in case
\[ Dl^\gamma = 0 \Rightarrow \lambda_\gamma dx^\gamma = 0 \Rightarrow \delta \lambda_\gamma dx^\gamma + \lambda_\gamma dx^\gamma = 0 \]
and \( B \) reduces to the form \( B = -d(\mu_\gamma + 2l_\gamma) \Delta l^\gamma. \) Then
\[ A^\alpha_\beta r_\beta d\Delta l^\gamma = FC^\alpha_\gamma r_\beta (\delta^\beta A^\gamma_\delta - \partial_\beta A^\gamma_\delta \Gamma^\gamma_\delta) \] \( dx^\gamma \Delta l^\delta - \]
\[ \theta^\beta_\alpha r_\beta B + FC^\alpha_\gamma r_\beta (F^{-1} \delta \dot{x}^\delta + F^{-1} d\delta \dot{x}^\delta + d\delta \dot{x}^\delta F^{-1}). \tag{4.10} \]
We can add and substract $\delta dz^\delta$, to the last term of (4.10), where from (4.5) we have

$$\delta dz^\delta = -\delta F z^\delta dF - F \delta z^\delta dF - F z^\delta \delta dF - \partial \Gamma^* \chi [F(\partial_\beta - A_\beta \gamma) \Delta l^\gamma - F \delta z^\delta dF] d\gamma - \Gamma^* \delta \delta z^\delta.$$

Using the homogeneity condition of $\Gamma^* \chi$ in $\dot{z}$ (first degree) and the relation $C_{\alpha \beta \gamma} \delta z^\delta = 0$ (4.10) has the form

$$A_{\alpha \beta \gamma} r_{\beta} d\Delta l^\gamma = -FC_{\alpha \beta \chi} r_{\beta} \partial \Gamma^* \chi (\partial_\delta - A_\delta \gamma) -$$

$$\partial_\gamma A_{\delta \chi} + \partial_\delta A_{\gamma \chi} \Gamma^* \gamma + d \Gamma^* \gamma | d\gamma \Delta l^\delta - C_{\alpha \beta \gamma} r_{\beta} (\delta d - d\delta) \dot{z}^\delta -$$

$$C_{\alpha \beta \gamma} r_{\beta} \Gamma^* \gamma \delta d \gamma \gamma - \theta_{\alpha \beta} r_{\beta} B.$$

Substituting (4.11) into (4.9) using (3.6), (3.15), (4.8) and (2.9) we obtain

$$\Delta l^\delta - D^\delta = -Dr_{\alpha} - r_{\beta} [P_{\beta}^\delta \gamma - A_{\beta} \gamma \Gamma^* \gamma +$$

$$FC_{\alpha \beta \chi} (\partial_\delta - A_\delta \gamma) + \partial_\delta A_{\gamma \chi} \Gamma^* \gamma + 2^{-1} A_\delta \delta \lambda \lambda] d\gamma \Delta l^\delta - \theta_{\alpha} \beta r_{\beta} B.$$

In the non recurrent Finsler space $F_n$ when $\theta_{\alpha} \beta = 0$ (4.12) reduces to the form

$$(\Delta l^\delta - D^\delta) r_{\alpha} = -Dr_{\alpha} - r_{\beta} P_{\alpha}^\beta \gamma \delta d \gamma \gamma \Delta l^\delta.$$

When $Dr_{\alpha} = 0$, $\Delta r_{\alpha} = 0$ from (4.13) it is easy to see that

$$Dr_{\alpha} = -r_{\beta} P_{\alpha}^\beta \gamma \delta d \gamma \gamma \Delta l^\delta.$$

**Case 3.** Let us consider the cycle of lintements

$$P(x, \dot{x}),$$

$$P_1 (x + dx, \dot{x} + d\dot{x}), \quad Dl^\delta = 0 \Rightarrow dz^\delta = \dot{x}^\delta dF - \Gamma^* \gamma \delta dx^\beta,$$

$$P_2 (x + dx, \dot{x} + d\dot{x}), \quad \Delta l^\delta = 0 \Rightarrow \delta \dot{z}^\delta = \dot{x}^\delta dF - \Gamma^* \beta \gamma d\dot{z}^\beta,$$

$$P_3 = P_1 + \delta P_1 = (x + dx + \delta x, \dot{x} + d\dot{x}), \quad P_3 = P_2 + dP_2 = (x + dx + \delta x, \dot{x} + d\dot{x} + d\delta \dot{x}),$$

$$\text{From } Dr_{\alpha} = dr_{\alpha} - \Gamma^* \beta \gamma r_{\beta} d\gamma \gamma \text{ it follows }$$

$$(\Delta l^\delta - D^\delta) r_{\alpha} = (\delta \delta - d\delta) r_{\alpha} - \Gamma^* \beta \gamma r_{\beta} (\delta \delta - d\delta) \dot{z}^\gamma - 2^{-1} K_{\alpha} \beta \gamma \delta [d\gamma \delta z^\delta]$$

$$\text{From } (4.14), (4.15) \text{ and } C_{\alpha} \beta \gamma \dot{z}^\gamma = 0 \text{ it follows }$$

$$C_{\alpha} \beta \gamma (\delta \delta - d\delta) \dot{z}^\gamma = C_{\alpha} \beta \gamma (\delta \delta - d\delta) \dot{z}^\gamma - 2^{-1} FC_{\alpha} \beta K_\beta \gamma \delta [d\gamma \delta z^\gamma].$$

$$\text{From } (4.17) \text{ and } (2.9) \text{ we obtain }$$

$$(\Gamma^* \beta \gamma (\delta \delta - d\delta) z^\gamma) = (\Gamma^* \beta \gamma - \theta \delta \gamma \lambda \gamma) (\delta \delta - d\delta) \dot{z}^\gamma +$$

$$C_{\alpha} \beta \gamma (\delta \delta - d\delta) \dot{z}^\gamma + 2^{-1} FC_{\alpha} \beta K_\beta \gamma \delta [d\gamma \delta z^\gamma].$$
Substituting (4.18) into (4.16) and using (3.1) we get

\[(\Delta D - D\Delta) r_\alpha = -\overline{Dr}_\alpha - 2^{-1}(K_\alpha^{\beta\gamma\delta} + FC_\alpha^{\beta\gamma} K_\gamma^\delta) [dx^\gamma \delta x^\delta] + \theta_\alpha^{\beta\gamma} \lambda_\gamma (\delta d - d\delta) x^\gamma.\]

For the case of a non recurrent Finsler space (when \(\lambda_\gamma = 0, \mu_\gamma = 0\)) and \(\theta_\alpha^{\beta\gamma} = 0\) \(\Gamma_\alpha^{\beta\gamma}\) and \(A_\alpha^{\beta\gamma} = FC_\alpha^{\beta\gamma}\) are the Cartan connection coefficients. In this case for \(Dr_\alpha = 0\) and \(\Delta r_\alpha = 0\) we obtain.

\[\overline{Dr}_\alpha = -2^{-1} R_\alpha^{\beta\gamma\delta} r_\beta [dx^\gamma \delta x^\delta]\]

where \(R_\alpha^{\beta\gamma\delta} = K_\alpha^{\beta\gamma\delta} + A_\alpha^{\beta\gamma} K_\gamma^\delta.\)

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(Received 16 02 1987)