MERCERIAN THEOREMS FOR BEEKMANN MATRICES

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To the memory of B. Martić

Abstract. A matrix $A = (a_{nk})$ is called normal if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n$. Such a matrix has a normal inverse $A^{-1} = (a_{nk})$. If the inverse $A^{-1}$ of a normal and regular matrix $A$ satisfies the conditions $a_{nk} \leq 0$ for $k < n$ and $a_{nn} > 0$ for all $n$, we call such a matrix a Beekmann matrix. Beekmann introduced those matrices and proved that for such a matrix $A$, the matrix $B = (I + \lambda A)/(1 + \lambda)$ is Mercerian for $\lambda > -1$. (I is the identity matrix.)

This paper extends Beekmann's theorem to the case of $R^\beta$-Mercerian matrices, $\beta > 0$.

1. Let $A = (a_{nk})$ be a normal matrix, i.e., such that

$$a_{nk} = 0 \text{ for } k > n \quad \text{and} \quad a_{nn} \neq 0 \text{ for all } n. \tag{1.1}$$

Such a matrix has a normal inverse $A^{-1} = (a_{nk})$, so that the transformations

$$y_n = \sum_{k=1}^{n} a_{nk} x_k \ldots, \quad n = 2, \ldots \tag{1.2}$$

and

$$x_n = \sum_{k=1}^{n} a_{nk} y_k \ldots, \quad n = 1, 2, \ldots \tag{1.3}$$

are inverse one to the other.

If the inverse $A^{-1}$ of a normal and regular matrix $A$ satisfies the conditions

$$a_{nk} \leq 0 \text{ for } k < n \quad \text{and} \quad a_{nn} > 0 \text{ for all } n, \tag{1.4}$$

we shall call such a matrix a Beekmann matrix.

Beekmann introduced those matrices in [1] and proved that for such a matrix $A$, the matrix $B = (I + \lambda A)/(1 + \lambda)$ is Mercerian for $\lambda > -1$. (I is the matrix.)

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The aim of this paper is to extend Beekmann’s theorem to the case of $R$-Mercerian matrices, $\beta \geq 0$.

2. A sequence $s$ is said to be regularly varying iff

\[
\lim_{n \to \infty} \frac{s(x_n)}{s(n)} = h(t)
\]

exists for every $t > 0$. ($[x]$ is the greatest integer $\leq x$). Such sequences (and functions) were introduced by J. Karamata [2]; today they play an essential role in summability and probability. (1.5) implies that there is a real number $\beta$ such that $h(t) = t^\beta$. The number $\beta$ is called the order of $s$. In addition, a regularly varying sequence of “order 0” (i.e., for which the limit in (1.5) equals 1) is called a slowly varying sequence. It can be proved [2] that every regularly varying sequence $s$ of order $\beta > 0$ can be written in the form

\[
s_n = n^\beta L(n),
\]

where $L$ is a slowly varying sequence.

By $R_\beta$, $\beta > 0$, we denote the class of regularly varying sequences of order $\beta$, and by $R_0$ the class of slowly varying sequences.

At last, we say that a matrix $A$ is $R_\beta$-regular ($\beta > 0$) iff for every $s \in R_\beta$ and any sequence $r$

\[
r_n \sim s_n \implies \sum_{k=1}^n a_{nk}r_k \sim s_n, \quad n \to \infty
\]

and it is called $R_\beta$-Mercerian iff

\[
\sum_{k=1}^n k = 1^n a_{nk}r_k \sim s_n, \quad n \to \infty
\]

(Obviously, a matrix $A$ is regular iff $r_n \to L$ implies $\sum_{k=1}^n a_{nk}r_k \to L$, and Mercerian iff $\sum_{k=1}^n a_{nk}r_k \to L$ implies $r_n \to L$, $n \to \infty$).

3. The $R_\beta$-regularity theorems for matrices were first established by M. Vuilleumier in [6]. The first $R_\beta$-Mercerian theorems for regular, invertible triangular matrices were established by S. Zimering in [3].

Using their results, B. Martić [5] proved the following.

**Theorem M.** Let $A = (a_{nk})$ be normal, nonnegative (i.e. $a_{nk} \geq 0$) and regular matrix which, for some $\gamma > 0$, satisfies the condition

\[
\sum_{k=1}^n a_{nk}k^{-\gamma} = O(n^{-\gamma}), \quad n \to \infty
\]

Then the matrix $B = (I + \lambda A)/(1 + \lambda)$, where $I$ is the unit triangular matrix, is $R_0$-Mercerian for $|\lambda| < 1$. 
Martić supposed \( \sum_{k=1}^{n} a_{nk} = 1 \), but his proof is valid also in case \( \sum_{k=1}^{n} a_{nk} \rightarrow 1 \). Since, in case of a Beekmann matrix \( A \), the conditions (1.4) imply

\[
a_{nk} \geq 0 \text{ for all } k < n \text{ and } a_{nn} > 0,
\]

we can apply Martić’s theorem and obtain

**Lemma 3.1.** If a Beekmann matrix \( A \) satisfies the condition (3.1) for some \( \gamma > 0 \), then the matrix \( B = (I + \lambda A)/(1 + \lambda) \) is \( R_{0} \)-Mercerian for \( \vert \lambda \vert < 1 \).

Lemma 3.1 reduces the proof of a general \( R_{0} \)-Mercerian theorem for Beekmann matrices to the case \( \lambda \geq 1 \). However, a method used by Tanović-Miller [4] and based upon the relations

\[
\beta_{nk} \leq 0 \text{ for } k < n \text{ and } \beta_{nn} > 0 \text{ for all } n,
\]

\[
\sum_{k=1}^{n} \beta_{nk} \rightarrow 1, \ n \rightarrow \infty
\]

and

\[
\sum_{k=1}^{n} \vert \beta_{nk} \vert^{1-\gamma} = O(n^{-\gamma}), \ n \rightarrow \infty
\]

for the inverse \( B^{-1} \) of \( B \) above supplies readily the proof in this case. Tanović-Miller considered non-negative, normal, normalized \( (\sum_{k=1}^{n} a_{nk} = 1) \) matrices \( A \), which satisfy the conditions

\[
a_{n1} > 0, \ a_{n+1,i}a_{nk} \leq a_{ni}a_{n-1,k},
\]

for \( 1 \leq k \leq i \leq n - 1 \) and the condition (3.1). and from these derived (3.3)-(3.5). Once one has (3.3)-(3.5), the proof is a straightforward application of Theorem 4.1 of M. Vuillemin in [6].

Thus, if we prove that for a Beekmann matrix \( A \), which satisfies (3.1), the inverse \( B^{-1} \) of \( B = (I + \lambda A)/(1 + \lambda) \) satisfies (3.3)-(3.5) for \( \lambda > 1 \), Lemma 3.1 will be completed for all \( \lambda > -1 \).

4. Our main result is contained in

**Theorem 4.1.** If \( A \) is a Beekmann matrix and \( B = (I + \lambda A)/(1 + \lambda) \), then \( B \) is a Beekmann matrix for \( \lambda > 0 \).

*Proof:* Let \( A = (a_{nk}) \), \( A^{-1} = (a_{nk}) \), \( B = (b_{nk}) \) and \( B^{-1} = (\beta_{nk}) \).

Let us remark that the transformations

\[
y_{n} = \sum_{k=1}^{n} b_{nk}x_{k}
\]
and

$$x_n = \sum_{k=1}^{n} \beta_{nk} y_k$$

are inverse.

Since $b_{nk} = \lambda a_{nk}/(I + \lambda)$ for $k < n$ and $b_{nn} = (1 + \lambda a_{nn})/(1 + \lambda)$, $b_{nk} = 0$ for $k > n$, $B$ is normal and obviously regular. Thus $B^{-1}$ exists and it is normal. Moreover, (4.1) and (4.2) are inverse and (1.2) and (1.3) are inverse.

The case $\lambda = 0$ being trivial, let $\lambda > 0$, and let $\varepsilon = (1 + \lambda)/\lambda$. Obviously, $\varepsilon > 1$.

We have for any sequence $x$,

$$\sum_{k=1}^{n} \alpha_{nk} x_k = \varepsilon b_{nk} - (\varepsilon - 1)x_n;$$

introducing the sequence $y$, defined by (4.1), this gives

$$\sum_{k=1}^{n} \alpha_{nk} x_k = \varepsilon y_n - (\varepsilon - 1)x_n.$$  

If in (1.2) we replace $y_n$ by $\varepsilon y_n - (\varepsilon - 1)x_n$ and use (1.3), from (4.3) we obtain

$$x_n = \varepsilon \sum_{k=1}^{n} \alpha_{nk} y_k - (\varepsilon - 1) \sum_{k=1}^{n} \alpha_{nk} x_k$$

which, using in the second sum on the right side formula (4.2), yields, after some elementary computations,

$$x_n = \sum_{k=1}^{n} \left\{ \varepsilon \alpha_{nk} - (\varepsilon - 1) \sum_{i=k}^{n} \alpha_{ni} \beta_{ik} \right\} y_k.$$  

From this and (4.2) we obtain at once

$$\beta_{nk} = \varepsilon \alpha_{nk} - (\varepsilon - 1) \sum_{i=k}^{n} \alpha_{ni} \beta_{ik},$$

and, in particular, for $k = 1, 2, \ldots, n$,

$$\beta_{kk} = \{\varepsilon/(1 + (\varepsilon - 1)\alpha_{kk})\} \alpha_{kk}$$

and for $k \geq 2$

$$\beta_{k,k-1} = \varepsilon \alpha_{k,k-1}/(1 + (\varepsilon - 1)\alpha_{kk})(1 + (\varepsilon - 1)\alpha_{k-1,k-1}).$$

Now, solving (4.4) for $\beta_{nk}$ and using (4.5) we obtain, for $k = 1, 2, \ldots, n - 2$

$$\beta_{nk} = \frac{\varepsilon}{(1 + (-1)\alpha_{nn})(1 + (\varepsilon - 1)\alpha_{kk})} \alpha_{nk} - \frac{\varepsilon - 1}{1 + (\varepsilon - 1)\alpha_{nn}} \sum_{i=k+1}^{n-1} \alpha_{ni} \beta_{ik}.$$
Since \( \alpha_{kk} > 0 \) and \( \alpha_{k,k-1} \leq 0 \) we conclude from (4.5) and (4.6) (with \( k = n \)) that \( \beta_{nn} > 0 \) for all \( n \) and \( \beta_{n,n-1} \leq 0 \), for \( n \geq 2 \). Then, from (4.7) we conclude: if \( \beta_{k+1,k}, \beta_{k+2,k}, \ldots, \beta_{n-1,k} \) are all \( \leq 0 \) for \( k < n \), then \( \beta_{nk} \leq 0 \) too, for \( k = 1, 2, \ldots, n - 2 \), which completes the proof of the theorem.

**Corollary 4.1.1.** Let \( A \) be a Beekmann matrix which, for some \( \gamma > 0 \), satisfies the condition (3.1). Then \( B^{-1} \), the inverse of \( B = (I + \lambda A)/(1 + \lambda) \), satisfies the condition (3.5) for \( \lambda \geq 0 \).

**Proof.** We use notations of Theorem 4.1. If \( D \) is any matrix, by \((D)_{nk}\) we denote its element in \( n \)-th row and \( k \)-th column. \( \delta^k_n \) denotes the Kronecker symbol (= 1 if \( k = n \), 0 otherwise).

Since
\[
\sum_{i=1}^{n} b_{ni} \beta_{ik} = (BB^{-1})_{nk} = \delta^k_n,
\]
we have, for \( k < n \),
\[
\sum_{i=1}^{n-1} b_{ni} \beta_{ik} = -b_{nn} \beta_{nk},
\]
and, since \( \beta_{nn} = 1/b_{nn} \),
\[
(4.8) \quad -\beta_{nk} = \beta_{nn} \sum_{i=1}^{n-1} b_{ni} \beta_{ik}.
\]

Taking into account the relations \( \beta_{ik} \leq 0 \) for \( i \neq k \) (\( B \) is Beekmann, by Theorem 4.1), \( b_{ni} \geq 0 \) and \( \beta_{kk} > 0 \), we obtain from (4.8), for \( k < n \)
\[
(4.9) \quad -\beta_{nk} \leq \beta_{nn} b_{nk} \beta_{kk} \leq b_{nk} (1 + \lambda)^2.
\]

since
\[
\beta_{nn} \beta_{kk} = \frac{1 + \lambda}{1 + \lambda a_{nn}} \cdot \frac{1 + \lambda}{1 + \lambda a_{kk}} \leq (1 + \lambda)^2.
\]

Using the relations between the elements of \( A \) and \( B \), the fact that \( B \) is Beekmann, (3.1) and (4.9), we have:
\[
\sum_{k=1}^{n} |\beta_{nk}| k^{-\gamma} = \sum_{k=1}^{n-1} -\beta_{nk} k^{-\gamma} + \beta_{mn} n^{-\gamma} \leq (1 + \lambda)^2 \sum_{k=1}^{n-1} b_{nk} k^{-\gamma} + \frac{1 + \lambda}{1 + \lambda a_{nn}} n^{-\gamma}
\]
i.e.
\[
\sum_{k=1}^{n} |\beta_{nk}| k^{-\gamma} \leq \lambda (1 + \lambda) \sum_{k=1}^{n-1} a_{nk} k^{-\gamma} + O(n^{-\gamma}),
\]
which, by (3.1), gives (3.5).

**Corollary 4.1.2.** The matrix \( B^{-1} \) of Theorem 4.1 satisfies (3.4).
Proof. From (4.9) follows

\[ |\beta_{nk}| \leq (1 + \lambda)^2 b_{nk}, \ k < n \]

i.e., (since \( B \) is regular) for every fixed \( k \), \( |\beta_{nk}| \to 0, \ n \to \infty \).

Also, by the same inequality and the fact that

\[
\beta_{nn} = 1/b_{nn} = \frac{1 + \lambda}{1 + \lambda a_{nn}} \sum_{k=1}^{n} |\beta_{nk}| < (1 + \lambda)^2 \sum_{k=1}^{n-1} b_{nk} + \frac{1 + \lambda}{1 + \lambda a_{nn}}
\]

and since \( B \) is regular, there is \( M > 0 \) such that

\[
(4.10) \quad \sum_{k=1}^{n} |\beta_{nk}| \leq M.
\]

Set now in (4.1) \( x_k = 1 \) for all \( k \), so that \( y_n = \sum_{k=1}^{n} b_{nk} \). Then, by (4.2)

\[
1 = \sum_{k=1}^{n} \beta_{nk} y_k
\]

and so

\[
1 - \sum_{k=1}^{n} \beta_{nk} = \sum_{k=1}^{n} \beta_{nk} (y_k - 1).
\]

Since \( y_k - 1 \to 0, \ k \to \infty \), by (4.10) and the fact that, for fixed \( k \), \( |\beta_{nk}| \to 0, \ n \to \infty \) follows \( \lim_{n \to \infty} \sum_{k=1}^{n} \beta_{nk} = 1 \) in usual way.

Remark. A consequence of the content of Corollary 4.1.2 is that \( B^{-1} \) is a regular matrix. Contrary to this, \( A^{-1} \) does not need to be regular. For example, for the matrix \( A = (1/n)_{k \leq n} \) of arithmetic means, \( a_{nk} = 0 \) for \( k \leq n - 2, \ a_{n,n-1} = -(n-1), \ a_{nn} = n \) and \( \sum_{k=1}^{n} |a_{nk}| = 2n - 1 \) is not bounded!

5. We are able now to prove the extensions of Beekmann’s Mercerian Theorem to regularly varying functions.

Theorem 5.1. Let \( A \) be Beekmann matrix, such that, for some \( \gamma > 0 \),

\[
(5.1) \quad \sum_{k=1}^{n} a_{nk} k^{-\gamma} = O(n^{-\gamma}), \ n \to \infty.
\]

Then, for \( \lambda > -1 \), the matrix \( B = (I + \lambda A)/(1 + \lambda) \) is \( R_0 \)-Mercerian.

Proof. Case \( |\lambda| < 1 \) by Lemma 3.1. For \( \lambda \geq 1 \), by Theorem 4.1 and its Corollaries, \( B^{-1} \), the inverse of \( B \), satisfies all the conditions (3.3) – (3.5). By the remark at the end of section 3, \( B \) is \( R_0 \)-Mercerian.
Since every regularly varying sequence \( s \) of order \( \beta > 0 \), satisfies (1.6), applying Theorem 5.1 to the sequence \( \{s_n/n^3\} \), we obtain, in a similar way as Martić in [5],

**Theorem 5.2.** Let \( A \) be a Beekmann matrix such that there are two numbers \( \alpha \) and \( \beta \), \( 0 < \alpha < \beta \), for which

\[
\sum_{k=1}^{n} a_{nk} \left( \frac{k}{n} \right)^{\alpha} \rightarrow A_\alpha, \quad \text{and} \quad \sum_{k=1}^{n} a_{nk} \left( \frac{k}{n} \right)^{\beta} \rightarrow A_\beta, \quad n \to \infty.
\]

Then, for every \( \lambda \) such that \( 1 + \lambda A_\alpha > 0 \) and \( 1 + \lambda A_\beta > 0 \), the matrix \( B_\beta = (I + \lambda A)/(1 + \lambda A) \) is \( R_\beta \)-Mercerian.

One should remark that conditions \( 1 + \lambda A_\alpha > 0 \) and \( 1 + \lambda A_\beta > 0 \) imply one another, depending on the sign of \( \lambda \).

**References**


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