ON CERTAIN CONDITIONS WHICH REDUCE
A FINSLER SPACE OF SCALAR CURVATURE TO
A RIEMANNIAN SPACE OF CONSTANT CURVATURE

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Abstract. We give certain conditions which reduce a Finsler space of scalar curvature to
a Riemannian space of constant curvature.

1. Preliminaries. Let $F_n$ be an $n$-dimensional Finsler space with the
fundamental functional $L(x, y)$, the positive definite metric tensor $g_{ij} = 1/2\delta_i^j L^2$
and the angular metric tensor $h_{ij} = g_{ij} - l_i l_j$, where $l_i = \delta_i^j L$, $\delta_i = \delta/\delta y^i$.

For a Cartan connection $\Gamma$, $h$- and $\nu$-covariant derivatives of a finsler tensor
field $X_j^i$ are denoted by $X_j^i$ and $X_j^i$. The $h$, $h \nu$- and $\nu$-curvature tensors of $\Gamma$
are $R_{hjk}^i$, $P_{hjk}^i$ and $S_{hjk}^i$ and the $(\nu)$ $h$, $(h)$ $h \nu$- and $(\nu)$ $h \nu$-torsion tensors of $\Gamma$
are $R_{ijk}^h$, $C_{ijk}^h$ and $P_{ijk}^h$ respectively. On the otherhand $H_{ijk}^h$ and $H_{ijk}^h$
tensors and $(\nu) h$-torsion, tensors of Berwald connection $B_T$ respectively.

The following relations are well known [4]:

\begin{align}
(1.1) & \quad H_{ijk}^h = \delta_i^j H_{jk}^h \\
(1.2) & \quad P_{ijk} = C_{ijk}^h \nu,
\end{align}

where the index $\nu$ stands for transvection by $y$ and $C_{ijk} = 1/2\delta_i^j g_{ij}$

\begin{align}
(1.3) & \quad H_{ijk}^h = H_{ijk}^h = R_{ijk}^h = R_{ijk}^h, \\
(1.4) & \quad H_{ijk}^h = R_{ijk}^h - C_{ij}^h P_{ijk} + \{P_{ijk} - P_{ij}^h P_{jk}^h - j | k \},
\end{align}

where $j | k$ means interchange of indices $j$, $k$ in the foregoing terms.

A hypersurface of $F_n$ defined by the equation $L(x, y) = 1$, where the point
$x = (x^i)$ is fixed and $y^i$ are variables, is called indicatrix. We denote by $p$
the projection of the tensor of the Finsler spaces on the indicatrix. For example, the

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projection of the tensor $T^i_j$ of type $(1,1)$ of $F_n$ on the indicatrix is $p \cdot T^i_j = h^i \cdot T^i_j \cdot h^i_j$, where $h^i = \delta^i_a - l^i_l a, l^i = g^i h^i_j = L^{-1} y^i$. A tensor $T$ satisfying $p \cdot T = T$ is called an indicatrix tensor. We have

\begin{align}
(1.5) \quad & a) \ p \cdot t^i = p \cdot t_i = 0, & b) \ p \cdot \delta^i_j = h^i_j, \\
& c) \ p \cdot \delta^i_k h^j_i = p \cdot h^j_i \big|_k = 0, & d) \ p \cdot \delta^i_k h^i_j = 2 C_{ijk}.
\end{align}

2. A Finsler space of scalar curvature. A Finsler space of scalar curvature is characterized by [6] any one of the following equations:

\begin{align}
(2.1a) \quad & H^i_j \equiv L(Kl_j + K_j/3)h^i_k - j \mid k, \\
& H^i_{hjk} = \{t_h(Kl_j + K_j/3) + K^h_{hj} + 2K^h_{hj}/3 + K_{hj}/3\}h^i_k \\
(2.1b) \quad & + l^i(Kl_k + K_k/3)h^j_h + h^i_k h^i_j K_k/3 - j \mid k
\end{align}

where $K_k = L \delta k K, K_{hj} = L p \cdot \delta_k K = K_{jh}$. Specially, if the scalar $K$ is constant, then the space is called a Finsler space of constant curvature.

**Proposition 2.1.** A Finsler space $F_n(n \geq 3)$ of scalar curvature $K$ satisfies

\begin{equation}
K_{ijk} + K^h_{hjk} - i \mid j = 0,
\end{equation}

where $K_{ijk} = L p \cdot \delta_i K_{jk}$.

**Proof.** From (1.5) and (2.1b), we have

$$L p \cdot \delta_m H^i_{hjk} = (h^i_m K_j/3 + K^m h^i_j + 2LK^h_{hjm} + 2K^h_{hj}/3 + K^l_{hj}/3)h^i_k$$
$$+ h^i_m K_k h^i_j/3 + h^i_k h^i_j K_k/3 - j \mid k$$

Considering the skew-symmetric part of the above equation in the indices $h$ and $m$ and using the fact $\delta_m H^i_{hjk} = \delta_k H^i_{hjk}$, we get

$$[(K^m h^i_j + 2K^l h^i_j/3 + K^i_{m hj}/3)h^i_k - j \mid k] - h \mid m = 0$$

which is simplified as

\begin{equation}
[(K^m h^i_j + K^m_{hj})h^i_k - j \mid k] - h \mid m = 0
\end{equation}

Contracting (2.3) in indices $i$ and $k$, we get (2.2).

**Remark 2.1.** Proposition 2.1, and the definition of $K_j, K_{hj}$ and $K_{ijk}$ imply that when any one of them is zero, then the other two are automatically zero. $K_j = 0$ means that $K$ is independent of $y$. Thus $K$ is constant (Matsumoto [4, Prop. 26.1]). If for a Finsler space $F_n$ of scalar curvature any one of the tensors $K_i, K_{hj}$ and $K_{ijk}$ vanishes, $F_n$ is of constant curvature.

**Proposition 2.2.** A Finsler space $F_n$ of scalar curvature $K$ with $P_{hij0} = 0$ satisfies

\begin{equation}
h_{ik}(3K K_{jm} - K_{jK_m}) + h_{jk}(3K K_{im} - K_{kK_m}) - h \mid m = 0
\end{equation}
Proof. A Finsler space $F_n$ of scalar curvature $K$ satisfies [7]
\begin{equation}
L^{-1} P_{hij0} + L K C_{hij} + (K_h h_{ij} + K_i h_{hj} + K_j h_{hi})/3 = 0.
\end{equation}
Since, $P_{hij0} = 0$, (2.5) leads to
\begin{equation}
L K C_{hij} + (K_h h_{ij} + K_i h_{hj} + K_j h_{hi})/3 = 0.
\end{equation}
Differentiating the equation above partially with respect to $y^m$ and applying $p$ to the resulting equation and using (1.5) we get
\begin{align*}
3 L K C_{hij} + 3 L^2 K p \cdot \delta_m C_{hij} + (2 L C_{ijm} K_h + h_{ij} K_{hm} + 2 L C_{ijm} K_i \\& h_{jm} K_j + h_{ij} K_{jm}) &= 0.
\end{align*}
Considering skew symmetric part of the above equation in the indices $h$ and $m$, we get
\begin{equation}
L C_{hij} K_m + h_{jh} K_{im} + h_{hi} K_{jm} - h | m = 0.
\end{equation}
By virtue of (2.6) and (2.7), we obtain (2.4).

A Riemannian space is characterized by [4]:
\begin{equation}
C_{hij} = 0.
\end{equation}

**Theorem 2.3.** A Finsler space $F_n$ of non-vanishing scalar curvature $K$ with $P_{hij0} = 0$ is a Riemannian space of constant curvature if
\begin{equation}
3 K K_{ij} - K^{m} K_{m} = 0,
\end{equation}
where $K^i = g^{im} K_{mj}$, $K^i = g^{im} K_{m}$.

Proof. Transvecting (2.4) by $h^h = g^{ih} - i h^i$ we get
\begin{align*}
(n - 1)(3 K K_{jm} - K_j K_m) - (3 K K^* - K^* K^*_j) h_{jm} &= 0
\end{align*}
which leads to
\begin{equation}
3 K K_{jm} - K_j K_m = 0
\end{equation}
because of (2.9).

Differentiating (2.10) partially with respect to $y^h$ and applying $p$ to the resulting equation, we have
\begin{equation}
3 K_h K_{jm} + 3 K K_{hjm} - K_h j K_m - K_j K_{hm} = 0
\end{equation}
Equations (2.10) and (2.11) give $K_m K_h K_j + 9 K^2 K_{hjm} = 0$ which yields
\begin{equation}
K_{hjm} - h | j = 0 \quad K \neq 0,
\end{equation}
By virtue of (2.2) and (2.12), we get $K_h h_{jm} - h | j = 0$ which shows that
\begin{equation}
K_h = 0.
\end{equation}
On account of remark 2.1 and equations (2.6), (2.8) and (2.13), we have the theorem.

**Corollary 2.4.** A Finsler space $F_n$ of non-vanishing constant curvature $(K_j = 0, K \neq 0)$ with $P_{hj|0} = 0$ is a Riemannian space of constant curvature.

**Proof.** Since $F_n$ is of constant curvature, we get $K_j = K_{hj} = 0$. Thus all the conditions of theorem 2.3 are fulfilled. Hence the corollary.

The $h$-curvature tensor of Rund connection is defined as follows [4]:

$$K^i_{hjk} = R^i_{hjk} - C^i_{hr}R^r_{jk}. \tag{2.14}$$

**Theorem 2.5.** A Finsler space $F_n(n \geq 3)$ of non-vanishing scalar curvature $K$ is a Riemannian space of constant curvature if the $h$-curvature tensor of Berwald and Rund coincide.

**Proof.** From (1.4) and (2.14), we obtain $P^h_{ij|k} - P^h_{ji}P^r_{k} - j | k = 0$ which implies $P_{hijk} - P_{hj}P^r_{ki} - j | k = 0$. Considering symmetric part of the above equation in $i$ and $h$, we have

$$P_{thjk} - \sum| k = 0 \tag{2.15}$$

Also from (1.4), we get

$$H_{thjk} + H_{hijk} = -2C_{ih}R^r_{jk} + 2(P_{thj} - j | k) \tag{2.16}$$

Substitution of (2.15) in (2.16) gives

$$H_{thjk} + H_{hijk} = -2C_{hi}R^r_{jk} \tag{2.17}$$

By virtue of (2.1a) and (2.1b), we obtain

$$p \cdot H^i_{jk} = LK_j h^i_k / 3 - j | k \tag{2.18a}$$
$$p \cdot H_{hijk} = (Kh_j + K_{hj} / 3)h_i - j | k \tag{2.18b}$$

Applying indicatric projection $p \cdot$ on (2.17) and using (2.18a) and (2.18b) we get

$$K_{ij}h_{hk} + K_{hj}h_{ik} - j | k = -2LK_jC_{hi} - j | k. \tag{2.19}$$

Since $P_{thj|0} = 0$ because of (2.15), using (2.5) and (2.19), we have

$$3K_{K_{ij}} - 2K_jK_j h_{hk} + 3K_{K_{hj}} - 2K_hK_j h_{ik} - j | k = 0 \tag{2.20}$$

(2.4) and (2.20) lead to $K_iK_j h_{hk} + K_hK_j h_{ik} - j | k = 0$. Transvecting the last relation by $h^h$, we get

$$(n - 1)K_iK_j - KmK_{m}h_{ij} = 0 \tag{2.21}$$
Transvecting the relation above by $K^i K^j$, we obtain $(n - 2)K^m K_m K^k K_k = 0$, which implies $K^m K_m = 0$. Hence $K_i = 0$ identically. By definition $K_{h j} = 0$ also. Thus all the conditions of theorem 2.3 are satisfied. Hence the Theorem.

$T$-tensor $T_{hijk}$ is defined by [3]

$$T_{hijk} = LC_{hijk} + l_h C_{ijk} + l_i C_{hjk} + l_j C_{hi k} + l_k C_{h ijk}. \tag{2.22}$$

Ikeda [2] has proved that Finsler tensor field $u^i$ satisfies

$$u^j_{ij0} - u^i_{ij0} - u^j_{i j0} \tag{2.23}$$

**Theorem 2.6.** A Finsler space of non-vanishing scalar curvature with vanishing $T$-tensor and $P_{hij0} = 0$ is a Riemannian space of constant curvature if $C_{hijk}|_{0} = 0$.

**Proof.** Since $T = 0$, (2.2) implies

$$LC_{hijk} = -l_h C_{ijk} - l_i C_{hjk} - l_j C_{hi k} - l_k C_{h ijk}. \tag{2.24}$$

Differentiating (2.24) $h$-covariantly and transvecting the resulting equation by $y$ and using (2.23), we obtain

$$L(P_{hij} - C_{hijk}) = l_h P_{ijk} - l_i P_{hjk} - l_j P_{hi k} - l_k P_{h ijk}.$$ 

Differentiating the equation above $h$-covariantly once again and transvecting the resulting equation by $y$ and using (2.23), $P_{hij0} = 0$ and $C_{hijk}|_{0} = 0$, we obtain

$$P_{hijk} = 0. \tag{2.25}$$

From (2.15) - (2.21) and (2.25) we have $K_i = 0$ and consequently $K_{h j} = 0$.

Thus the theorem follows in light of Theorem 2.3.

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**References**


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