ON THE APPROXIMATION OF CONTINUOUS FUNCTIONS

Alexandru Lupaş

Abstract. We construct a sequence $(J_n)$ of linear positive operators defined on the space $C(K)$, $K = [a, b]$, with the properties a) $J_n(f) \in C(K)$ is a polynomial of degree $\leq n$; b) if $f \in C(K)$ then there exists a positive constant $C_0$ such that $\|f - J_n f\| \leq C_0 \cdot \omega(f; 1/n)$, $n = 1, 2, \ldots$, where $\| \cdot \|$ is the uniform norm and $\omega(f; \cdot)$ is the modulus of continuity; c) for $f \in C(K)$ there exists a $C_1 > 0$ such that $\|f(x) - (J_n f)(x)\| \leq C_1 \cdot \omega(f; \Delta_n(x))$, $x \in K$ where $\Delta_n(x) = \sqrt{(x - a)(b - x)/n + n^{-2}}$, $n = 1, 2, \ldots$; d) if $\Delta_n^*(x) = \sqrt{(x - a)(b - x)/n}$ and $(J_n^* f)(x) = (J_n f)(x) + \frac{b - x}{b - a}[f(a) - (J_n f)(a)] + \frac{x - a}{b - a}[f(b) - (J_n f)(b)]$, then for every continuous function $f : [a, b] \to R$ there exists a positive constant $C_2$ such that $\|f(x) - (J_n^* f)(x)\| \leq C_2 \cdot \omega(f; \Delta_n^*(x))$, $x \in [a, b]$, $n = 1, 2, \ldots$. In this manner are presented constructive proofs of the well-known theorems of Jackson [8], Timan [14] and Teljakovskii [13]. Likewise, some other approximation properties of the operators $(J_n)$ are investigated.

1. Introduction and definitions. Let $K$ be a compact interval of the real axis and denote by $C(K)$ the normed linear space of continuous real-valued functions on $K$. As usually, the space $C(K)$ is normed by means of the uniform norm, that is $\|f\| = \max_{x \in K} |f(x)|$, $f \in C(K)$. We will use the notation $\|f\|_K$ to indicate that the maximum is taken over $K$ whenever it is necessary to make it clear which interval the norm is taken over. Likewise, by $\Pi_n$ is denoted the linear space of polynomials, with real coefficients, of degree at most $n$.

For $f \in C(K)$ let $(p_n^* f)$, $p_n^* f \in \Pi_n$ be the sequence of polynomials of best approximation to $f$; more precisely

$$\|f - p_n^* f\| \leq \|f - p_n\|$$

---

AMS Subject Classification (1980): Primary 41A35; Secondary 41A36.
for all polynomials \( p_n, p_n \in \Pi_n \). It is known that the operator \( P_n^* : C(K) \to \Pi_n \)
which maps \( f \) into \( P_n^* f \) is not a linear transformation. At the same time, if \( f \in C(K) \) and \( \omega(f; \cdot) \) is modulus of continuity defined, for \( \delta \geq 0 \), by

\[
\omega(f; \delta) = \max_{|t - s| \leq \delta, t, s \in K} |f(t) - f(s)|,
\]

then according to the well-known theorem of Jackson ([6], [8], [10]) the sequence \( (P_n^* f) \) satisfies the inequalities

\[
||f - P_n^* f|| \leq C_0 \cdot \omega(f; 1/n), \quad C_0 \in (0, 1 + \pi^2/2], \quad n = 1, 2, \ldots.
\]

Several authors (see [2], [3], [7]) have constructed explicitly sequences of polynomials \( (A_n f) \) which have essentially the same degree of precision of approximation to \( f \), as \( P_n^* f \). These polynomials \( A_n f, n = 1, 2, \ldots, f \in C(K) \), have the properties:

i) the operator \( A_n : f \to C_n f \) is linear on \( C(K) \);

ii) \( A_n(C(K)) \subseteq \Pi_m(\Pi_n) \), \( m(n) \geq n \);

iii) there exists an interval \([ c, d ] \), \( a < c < d < b \), \( K = [a, b] \), such that for \( f \in C(K) \) and \( ||f - A_n f||_{[c, d]} \leq C \cdot \omega(f; 1/n), C > 0, n = 1, 2, \ldots \). Therefore, these kinds of polynomial operators \( A_n : C(K) \to \Pi_m(\Pi_n) \), \( n = 1, 2, \ldots \),
cannot be used to approximate on all of \( K = [a, b] \). They are only efficient on subintervals \([ c, d ] \) with \([ c, d ] \subseteq K \).

In 1951, Timan [14] has proved that if \( f \in C[a, b] \), then for every \( n \) there exists an algebraic polynomial \( \tau_n f \) of degree at most \( n \) such that for all \( x \in [a, b] \)

\[
|f(x) - (\tau_n f)(x)| \leq C_1 \cdot \omega(f; \sqrt{(x-a)(b-x)/n + n^{-2}}) \quad n = 1, 2, \ldots
\]

where \( C_1 \), is a positive constant. The characteristic peculiarity of this inequality is the improvement of the order of approximation near the endpoints in comparison to the usual Jackson theorem. This motivates the following:

**Definition.** A sequence of operators \( (J_n) \) defined on \( C(K), K = [a, b], \) is said to be of Jackson-type, if

a) \( J_n(C(K)) \subseteq \Pi_n, n = 1, 2, \ldots \);

b) \( J_n : C(K) \to \Pi_n \) is a linear positive operator;

c) for every \( f, f \in C(K) \), there exists a positive constant \( C_0 \) such that

\[
||f - J_n f|| \leq C_0 \cdot \omega(f; 1/n), \quad n = 1, 2, \ldots,
\]

where \( ||\cdot|| = ||\cdot||_K \);

d) if \( f \in C(K) \), then for all \( x \in [a, b] \) and \( n = 1, 2, \ldots \)

\[
|f(x) - (J_n f)(x)| \leq C_1 \cdot \omega(f; \sqrt{(x-a)(b-x)/n + n^{-2}}),
\]

\( C_1 \) being a positive constant.

Taking into account that we will be concerned with the approximation of continuous functions \( f : K \to R, K = [a, b], \) by elements from \( \Pi_n \), and since the space \( \Pi_n \) remains invariant under the transformation \( x = (2t - a - b)/(b-a), t \in [a, b] \), it suffices to carry out the analysis for the interval \([-1, 1] \). Throughout this paper, \( C \)
will denote positive constants which are, in general, different. Likewise, \( I \) denotes the interval \([-1, 1] \).
2. A quadrature formula. Let $C^{(j)}(I)$ be the linear space of all functions $f : I \to \mathbb{R}$ which have a continuous $j^{th}$ derivative on the interval $I$. In order to prove some identities we need the following proposition.

**Lemma 1.** Let $n$ be a natural number and $s = s(n) = 1 + [n/2]$. If $f \in C^{(n+1)}(I)$, then there exists a point $\theta = \theta(n, f)$, $\theta \in (-1, 1)$, such that

$$
\int_{-1}^{1} \frac{f(t)}{\sqrt{1 - t^2}} dt = \frac{2\pi}{n + 2} \left[ 1 - \frac{(-1)^n}{4} f(-1) + \sum_{k=1}^{s} f(x_{kn}) \right] + R_n(f)
$$

(1)

where

$$
R_n(f) = \frac{\pi}{2^{n+1}} \cdot \frac{f^{(n+2)}(\theta)}{(n + 2)!} \text{ and } x_{kn} = \cos \left( \frac{2k - 1 - n}{n + 2} \right)
$$

(2)

**Proof.** Let us suppose that $n$ is an even natural number, $n = 2m - 2$, $m \geq 1$. Then (1) may be written as

$$
\int_{-1}^{1} \frac{f(t)}{\sqrt{1 - t^2}} dt = \frac{\pi}{m} \sum_{k=1}^{m} f \left( \cos \left( \frac{2k - 1 - n}{2m} \right) \right) + R_{2m-2}(f)
$$

(3)

where $R_{2m-2}(f) = \frac{\pi}{2^{m-1}} \cdot \frac{f^{(2m)}(\theta)}{[2m]!}$, $\theta \in (-1, 1)$.

This is the Mehler-Hermite formula with remainder term [9, p. 111, (7.3.6)].

Now let $n$ be an odd natural number, $n = 2m - 1$. Then (1) is the same as

$$
\int_{-1}^{1} \frac{f(t)}{\sqrt{1 - t^2}} dt = \frac{\pi}{2m + 1} f(-1) + \frac{2\pi}{2m + 1} \sum_{k=1}^{m} f \left( \cos \left( \frac{2k - 1 - n}{2m + 1} \right) \right) + R_{2m-1}(f).
$$

$$
R_{2m-1}(f) = \frac{\pi}{2^m} \cdot \frac{f^{2m+1}(\theta)}{(2m + 1)!}, \quad \theta \in (-1, 1),
$$

which is a quadrature formula attributed to Bouzitat. We note that the remainder term $R_n(f)$ from (1) may be represented on the space $C(I)$ as

$$
R_n(f) = \pi 2^{-n-1} [\theta_1, \theta_2, \ldots, \theta_{n+3}; f]
$$

where $[\theta_1, \theta_2, \ldots, \theta_{n+3}]$ denotes the divided difference at a system of distinct points $\theta_1, \theta_2, \ldots, \theta_{n+3}$ from $I$ (see [11]-[12]).

3. A sequence of Jackson type operators. Let $w(t) = 1/\sqrt{1 - t^2}$, $t \in (-1, 1)$, and $L_p w, 1 \leq p \leq \infty$, be the class of measurable functions on $I$ which satisfy $||f||_p < \infty$, where

$$
||f||_p = \left( \int_{-1}^{1} |f(t)|^p w(t) dt \right)^{1/p}, \quad 1 \leq p < \infty,
$$

and $||f||_\infty$ is the sup-norm. Further, by $X$ we denote one of the following function spaces: $C(I)$ or $L_p w$. 
Also we use the following notation:

\[
T_m(x) = \cos m(\arccos x)
\]

\[
\varphi_n'(x) = a_n \cdot \frac{1 + T_{n+2}(x)}{(x - \cos \pi/(n+2))^2}, \quad \varphi_n' \in \Pi_n, \quad a_n = \frac{1}{\pi(n+2)} \sin^2 \frac{\pi}{n+2}
\]

\[
t_k(f) = \int_{-1}^{1} f(t)T_k(t)w(t)dt, \quad f \in X,
\]

\[
\omega_k = \frac{1}{t_k(T_k)} = \begin{cases} 2/\pi, & k = 1, 2, \ldots \\ 1/\pi, & k = 0. \end{cases}
\]

Functions from \(X\) can be expanded in terms of Chebyshev polynomials. Every \(f \in X\) has the expansion

\[
f(x) \sim \sum_{k=1}^{\infty} \omega_k t_k(f)T_k(x), \quad x \in I,
\]

where \(t_k(f)\) are the Chebyshev coefficients defined as above.

In order to try to give a simple and unified approach to the theory of approximation by algebraic polynomials on a compact interval, Butzer and Stens [4] have introduced the translation operator \(\tau_x\), \(x \in I\), defined on \(X\) by:

\[
(\tau_x f)(t) = 1/2 \cdot [f(xt + \sqrt{1 - x^2} \sqrt{1 - t^2}) + f(xt - \sqrt{1 - x^2} \sqrt{1 - t^2})], \quad t \in I.
\]

If \(f, g \in L^1_w\), then their convolution product is defined by means of the equality

\[
(f \ast g)(x) = \int_{-1}^{1} (\tau_x f)(t)g(t)w(t)dt.
\]

This convolution has the following properties [1]: if \(f, g, h \in L^1_w\), then \(f \ast g \in L^1_w\) and:

i) \(f \ast g = g \ast f\);  ii) \(f \ast (g \ast h) = (f \ast g) \ast h\); iii) \(t_k(f \ast g) = t_k(f)t_k(g)\);

iv) if \(f \in L^1_w\), \(g \in L^p_w\), \(1 \leq p < \infty\), then \(f \ast g \in L^p_w\) and \(||f \ast g||_p \leq ||f||_1 \cdot ||g||_p\).

Taking into account that for \(k \geq 1\)

\[
T_k(xt + \sqrt{1 - x^2} \sqrt{1 - t^2}) = T_k(x)T_k(t) + k^{-2}\sqrt{1 - x^2} \sqrt{1 - t^2}T'_k(x)T'_k(t)
\]

\[
T_k(xt - \sqrt{1 - x^2} \sqrt{1 - t^2}) = T_k(x)T_k(t) - k^{-2}\sqrt{1 - x^2} \sqrt{1 - t^2}T'_k(x)T'_k(t),
\]

it follows that \((\tau_x T_k)(t) = T_k(x)T_k(t)\). Therefore, if \(f \in X\) has the expansion (5), then

\[
(\tau_x f)(t) \sim \sum_{k=0}^{\infty} \omega_k t_k(f)T_k(x)T_k(t),
\]

\[
(f \cdot g)(x) \sim \sum_{k=0}^{\infty} \omega_k t_k(f)t_k(g)T_k(x).
\]
Before proving the main results we need the following simple proposition.

**Lemma 2.** Let \( \varphi_n^* \) be defined as in (4) and \( p(t) = At^2 + Bt + C \). Then

\[
\int_{-1}^{1} \varphi_n^*(t)p(t)w(t)dt = p \left( \cos \frac{\pi}{n+2} \right) + \frac{A}{n+2} \sin^2 \frac{\pi}{n+2}. 
\]  
(8)

Moreover

\[
\int_{-1}^{1} \frac{\varphi_n^*(t)}{\sqrt{1+t}}dt < \frac{\pi \sqrt{2}}{2n}, \quad \int_{-1}^{1} \varphi_n^*(t)dt \leq \frac{\pi}{n}. 
\]  
(9)

**Proof.** We first observe that

\[
\varphi_n^* \left( \cos \frac{(2k-1)\pi}{n+2} \right) = \begin{cases} 
(n+2)/2\pi, & k = 1 \\
0, & k = 2, 3, \ldots, n, 
\end{cases}
\]

\[
\varphi_n^*(-1) = \frac{1 + (-1)^n}{\pi(n+2)} \sin^2 \frac{\pi}{2(n+2)}.
\]

Now, let \( q \in \Pi_{2n+2} \) be defined by

\[
q(t) = \varphi_n^*(t)p(t) = c_n t^{n+2} + Q(t), \quad Q \in \Pi_{n+1}.
\]

It is easy to see that \( c_n = 2^n a_n^2 \), where \( a_n \) is defined as in (4). Using Lemma 1 we observe that

\[
R_n(q) = \frac{A}{n+2} \sin^2 \frac{\pi}{n+2},
\]

and from (1) we have

\[
\int_{-1}^{1} q(t)w(t)dt = \frac{2\pi}{n+2} q(x_1) + R_n(q) = p(x_1) + \frac{A}{n+2} \sin^2 \frac{\pi}{n+2}.
\]

If \( p_1(t) = \sqrt{1-t}, p_2(t) = \sqrt{1-t^2} \), then according to (8) we find

\[
\int_{-1}^{1} \varphi_n^*(t)w(t)dt = 1, \quad \int_{-1}^{1} \varphi_n^*(t)|p_1(t)|^2w(t)dt = 2 \cdot \sin^2 \frac{\pi}{2(n+2)}
\]

\[
\int_{-1}^{1} \varphi_n^*(t)|p_2(t)|^2w(t)dt = \frac{n+1}{n+2} \sin^2 \frac{\pi}{n+2}.
\]  
(10)

Let \( \Phi_n : C(I) \to R \) be the linear positive functional defined by

\[
\Phi_n(f) = \int_{-1}^{1} \varphi_n^*(t)f(t)w(t)dt, \quad f \in C(I).
\]

Since \( \Phi_n(e_0) = 1, e_0(t) = 1 \), we have \( |\Phi_n(f)|^2 \leq |\Phi_n(f')|^2 \). Therefore, we obtain

\[
\Phi_n(p_1) \leq \sqrt{2 \cdot \sin^2 \frac{\pi}{2(n+2)}} < \frac{\pi \sqrt{2}}{2n}, \quad \Phi_n(p_2) \leq \sqrt{\frac{n+1}{n+2} \sin^2 \frac{\pi}{n+2}} < \frac{\pi}{n}.
\]
If $t^*_{kn} = t_k(\phi^*_n)$, i.e. $\phi^*_n(x) = \sum_{k=0}^n t^*_{kn} \omega_k T_k(x)$, then from (8)

$$
t^*_{0n} = 1, \quad t^*_{1n} = \cos \frac{\pi}{n + 2}, \quad t^*_{2n} = \frac{2(n + 1)}{n + 2} \cos \frac{\pi}{n + 2} - \frac{n}{n + 2}.
$$

Next we consider the kernel $L_n : I \times I \to R$, where

$$
L_n(x, t) = \sum_{k=0}^n t^*_{kn} \omega_k T_k(x) T_k(t).
$$

Taking into account (7), we obtain $L_n(x, t) = (\tau_x \phi^*_n)(t)$, that is $L_n(x, t) \geq 0$ for $(x, t) \in I \times I$.

Using the kernel we define the linear positive operators $J_n : C(I) \to \Pi_n, n = 1, 2, \ldots$, by

$$
(J_n f)(x) = (\phi^*_n \ast f)(x) = \int_{-1}^1 L_n(x, t) f(t) w(t) dt.
$$

(11)

The main result of this section is the following:

**Theorem 1.** The sequences of operators $(J_n)$ defined in (11) is of Jackson type. If $f \in C(I)$, then

i) $|f(x) - (J_n f)(x)| \leq C \cdot \omega(f; \Delta_n(x)), \quad x \in I,$

where

$$
\Delta_n(x) = \sqrt{1 - x^2/n + n^{-2}}, \quad C \in (0, 1 + \pi \sqrt{2} + \pi^2/2); \quad (12)
$$

ii) $\|f - J_n f\| \leq C_1 \cdot \omega(f; 1/(n + 2)), C_1 \in (0, 8)$.

**Proof.** If

$$
z_1(t, x) = |x - tx - \sqrt{1 - x^2 \sqrt{1 - t^2}}|, \quad z_2(t, x) = |x - tx + \sqrt{1 - x^2 \sqrt{1 - t^2}}|, \quad (13)
$$

then it may be proved that for $(t, x) \in I \times I$

$$
z_j(t, x) \leq \Delta_n(x) Q_n(t), \quad j = 1, 2,
$$

where $Q_n(t) = 2n \sqrt{1 - t} + n^2(1 - t) = 2np_1(t) + n^2|p_1(t)|^2$.

From (9)–(10) we have

$$
k_n = 1 + \int_{-1}^1 \phi^*_n(t) Q_n(t) w(t) dt < 1 + \pi \sqrt{2} + \pi^2/2. \quad (14)
$$

On the other hand, if $f \in C(I)$, $(t, x) \in I \times I$, we have

$$
|f(x) - (\tau_x f)(t)| \leq 1/2 \cdot |f(x) - f(xt + \sqrt{1 - x^2 \sqrt{1 - t^2}})| + 1/2 \cdot |f(x) - f(xt - \sqrt{1 - x^2 \sqrt{1 - t^2}})| \leq \omega(f; \Delta_n(x) Q_n(t)).
$$
The well-known inequality \( \omega(f; \lambda \delta) \leq (1 + [\lambda]) \omega(f; \delta) \) makes it possible to write
\[
|f(x) - \tau_x f(t)| \leq (1 + Q_n(t)) \omega(f; \Delta_n(x)), \quad (t, x) \in I \times I,
\]
\( \Delta_n(x) \) being defined in (12). Using the commutativity of the convolution product, for \( f \in C(I) \) and \( x \in I \) we have
\[
|f(x) - (J_n f)(x)| = |f(x)(e_0 * \varphi_n^*)(x) - (f * \varphi_n^*)(x)|
\leq \int_{-1}^{1} |f(x) - (\tau_x f)(t)| \varphi_n^*(t) \omega(t) dt.
\]
In this manner, from (14)–(15) we obtain
\[
|f(x) - (J_n f)(x)| \leq k_n \cdot \omega(f; \Delta_n(x)) \leq C \cdot \omega(f; \Delta_n(x))
\]
where \( C \leq C_0, \ C_0 = 1 + \pi \sqrt{2} + \pi^2/2 \) and \( x \in I \). From \( \omega(f; \Delta_n(x)) \leq \omega(f; (1 + 1/n)/n) \leq 2 \cdot \omega(f; 1/n) \) it follows that for every \( x \in I : |f(x) - (J_n f)(x)| \leq 2C_0 \omega(f; 1/n) \). Therefore
\[
||f - J_n f|| = \max_{x \in I} |f(x) - (J_n f)(x)| \leq 2C_0 \omega(f; 1/n).
\]
A sharper inequality may be obtained in the formula way: if \( Q_x(t) = (x - t)^2 \), then from (8)–(11)
\[
(J_n Q_x)(x) = 4 \left[ x^2 + \frac{n + 1}{n + 2} (1 - 2x^2) \cos^2 \frac{\pi}{2(n + 2)} \right] \cdot \sin^2 \frac{\pi}{2(n + 2)},
\]
i.e. \( W_n = \max_{x \in I} |(J_n Q_x)(x)| = \frac{n + 1}{n + 2} \sin^2 \frac{\pi}{n + 2} < \frac{\pi}{(n + 2)^2} \).
It is well-known that for a positive linear operator \( J : C(I) \to C(I), J e_0 = e_0 \), the inequality
\[
||f - J f|| \leq (1 + W/\delta^2) \omega(f; \delta), \quad \delta > 0, \ f \in C(I), \ W = \max_{x \in I} |(J Q_x)(x)|
\]
is verified [5]. In our case, with \( \delta = \frac{\pi}{(n + 2)} \) we obtain
\[
||f - J_n f|| \leq 2 \cdot \omega(f; \pi/ (n + 2)) \leq 8 \cdot \omega(f; 1/(n + 2)).
\]
Next we investigate the local degree of approximation by means of the polynomial operators \( J_n^* : C(I) \to \Pi_n, n = 1, 2, \ldots \), where
\[
(J_n^* f)(x) = (J_n f)(x) + (1 - x) / 2 \cdot [f(-1) -(J_n f)(-1)]
+ (1 + x) / 2 \cdot [f(1) - (J_n f)(1)], \quad x \in I,
\]
\( J_n \) being defined in (11).

**Theorem 2.** If \( J_n^* : C(I) \to \Pi_n \) is defined as in (16), then for \( f \in C(I) \) there exists a positive constant \( C^* \) such that
\[
|f(x) - (J_n^* f)(x)| \leq C^* \omega(f; \sqrt{1 - x^2}/n), \quad x \in I, \ n = 1, 2, \ldots,
\]
Proof. Let us denote $\Delta_n^*(x) = \sqrt{1 - x^2}/n$, $(\varepsilon_n f)(x) = f(x) - (J_n f)(x)$ and suppose that $x \in I_2 = (-\sqrt{1 - n^{-2}}, \sqrt{1 - n^{-2}})$, i.e., $n^{-2} < \Delta_n^*(x)$. According to Theorem 1, for $x \in I_2$ we have

$$|f(x) - (J_n^* f)(x)| = |(\varepsilon_n f)(x) - (1 - x)/2 \cdot (\varepsilon_n f)(-1) - (1 + x)/2 \cdot (\varepsilon_n f)(1)|$$

$$\leq C \cdot \omega(f; \Delta_n^*(x)) + C \cdot \omega(f; n^{-2}) \leq C_0 \cdot \omega(f; 2\Delta_n^*(x)) + C_0 \cdot \omega(f; \Delta_n^*(x)).$$

More precisely

$$|f(x) - (J_n^* f)(x)| \leq 3C_0 \omega(f; \Delta_n^*(x)) \quad x \in I_2. \quad (17)$$

Next we suppose that $\Delta_n^*(x) \leq n^{-2}$, i.e., $x \in I_1 \cup I_3$ where

$$I_1 = [-1, -\sqrt{1 - n^{-2}}], \quad I_3 = [\sqrt{1 - n^{-2}}, 1].$$

If $z_1$, $z_2$ are defined as in (13), then for $(x, t) \in U = I_3 \times I$ we have

$$z_j(x, t) \leq \Delta_n^*(x)S_n(t), \quad j = 1, 2 \quad (18)$$

where $S_n(t) = 1 + n\sqrt{1 - t^2} = 1 + np_2(t)$. Indeed

$$z_j(x, t) \leq p_2(x)p_2(t) + |t|(1 - x) \leq p_2(x)p_2(t) + (1 - x^2) \leq \Delta_n^*(x)S_n(t).$$

From (9)

$$\bar{a}_n = 1 + \int_{-1}^{1} \varphi_n^*(t)S_n(t)\omega(t)dt < 2 + \pi.$$

At the same time, for $(x, t) \in U$

$$|(\tau x f)(t) - f(t)| \leq 1/2 \cdot \omega(f; z_1(x, t)) + 1/2 \cdot \omega(f; z_2(x, t))$$

which together with (18) implies

$$|(\tau x f)(t) - f(t)| \leq (1 + S_n(t))\omega(f; \Delta_n^*(x)).$$

Likewise, for $(x, t) \in U$

$$|(\tau - x f)(t) - f(-t)| \leq (1 + S_n(t))\omega(f; \Delta_n^*(x)).$$

Therefore, in case $x \in I_3$,

$$|(J_n f)(x) - (J_n f)(1)| = |(f \ast \varphi_n^*)(x) - (f \ast \varphi_n^*)(1)|$$

$$\leq \int_{-1}^{1} \varphi_n^*(t)(\tau x f)(t) - f(t)\omega(t)dt \leq \bar{a}_n \omega(f; \Delta_n^*(x))$$

and $|(J_n f)(-x) - (J_n f)(-1)| \leq \bar{a}_n \omega(f; \Delta_n^*(x))$. In other words there exists a $C_1 \in (0, 2 + \pi)$ such that for $x \in I_3$: $C_1 \omega(f; \Delta_n^*(x))$.
Let us suppose \( x \in I_3 \); from (19) and Theorem 1:
\[
|f(x) - (J_n^* f)(x)| = |f(x) - f(1)| - |(J_n f)(x) - (J_n f)(1)| \\
+ (1 - x)/2 \cdot |(\varepsilon_n f)(1) - (\varepsilon_n f)(-1)| \\
\leq \omega(f; 1 - x) + C_1 \omega(f; \Delta_n^*(x)) + (1 - x) C_0 \omega(f; n^{-2}) \\
\leq \omega(f; 1 - x^2) + C_1 \omega(f; \Delta_n^*(x)) + (1 - x^2) C_0 \omega(f; n^{-2}) \\
\leq (1 + C_1) \omega(f; \Delta_n^*(x)) + C_0 \Delta_n^*(x) \omega(f; n^{-2}).
\]
It is known that for \( 0 \leq \delta_1 \leq \delta_2 \) one has \( \delta_1 \omega(f; \delta_2) \leq 2 \delta_2 \omega(f; \delta_1) \). If we select \( \delta_1 = \Delta_n^*(x) \), \( \delta_2 = n^{-2} \), \( x \in I_3 \), then
\[
\Delta_n^*(x) \omega(f; n^{-2}) \leq 2n^{-2} \omega(f; \Delta_n^*(x))
\]
In conclusion, for \( x \in I_3 \):
\[
|f(x) - (J_n^* f)(x)| \leq (1 + C_1 + 2n^{-2} C_0) \omega(f; \Delta_n^*(x))
\]
that is
\[
|f(x) - (J_n^* f)(x)| \leq C^* \omega(f; \Delta_n^*(x)), \quad n = 1, 2, \ldots, \tag{20}
\]
with \( 0 < C^* < 5 + (1 + 2 \sqrt{2}) \pi + \pi^2 \). Using the second inequality from (19) it may be shown that (20) is verified for \( x \in I_1 \) too. Taking into account (17) we conclude that (20) is true for all \( x \in I \).

**Theorem 3.** Let \( J_n \) be defined as in (11) and \( x \) fixed in \( I \). Then to each function \( f \in C(I) \) corresponds a system \( \Theta_1, \Theta_2, \Theta_3 \) of distinct points from \( I \) such that
\[
(J_n f)(x) = f(c \cdot \cos \pi/(n + 2)) + V_n(x)[\Theta_1, \Theta_2, \Theta_3; f] \tag{21}
\]
where \( V_n(x) = \frac{n(1-x^2)+1}{n+2} \sin^2 \frac{\pi}{n+2} \).

**Proof.** In [11]-[12] it is proved that if \( (L_n) \) is a sequence of positive linear operators defined on \( C(K) \) and \( L_n e_0 = e_0, L_n e_k = a_k, e_k(t) = t^k \), then for \( f \in C(K) \) and \( x \in K \):
\[
(L_n f)(x) = f[a_1 n(x)] + [a_2 n(x) - a_1 n(x)] [\Theta_1, \Theta_2, \Theta_3; f] \tag{22}
\]
where \( \Theta_n = \Theta_{in}(f, x), i = 1, 2, 3 \), are distinct points from \( K \). In our case, taking into account that \( J_n T_k = t_k^* n T_k, k = 0, 1, 2 \), one finds
\[
a_1 n(x) = x \cdot t_n^* x = x \cdot \cos \pi/(n + 2),
\]
\[
a_2 n(x) = e_2(x) - \frac{1}{2} (1 - t_2 n) T_2(x) = x^2 + (1 - 2x^2) \frac{n+1}{n+2} \sin^2 \frac{\pi}{n+2},
\]
and (22) proves the theorem.

In the case when \( f \in C^{(2)}(I) \) the equality (21) makes it possible to show that the remainder-term may be written as
\[
f(x) - (J_n f)(x) = Z(n, f, x) \sin^2 \frac{\pi}{2(n+2)}
\]
where for \( x \) fixed in \( I \)
\[
Z(n, f, x) = 2 \left[ x f'(x_n) + \frac{n(1 - x^2) + 1}{n + 2} f''(x_n) \cos^2 \frac{\pi}{2(n + 2)} \right],
\]
\( x_n = x_n(f, x) \) being points from \( I = [-1, 1] \).

REFERENCES