SOME RELATIONS FOR GRAPHIC POLYNOMIALS

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Abstract. Let $G$ be a graph and $A$ and $B$ its two subgraphs with disjoint vertex sets. A number of results is obtained, relating the characteristic, matching and $\mu$-polynomials of $G$, $G-A$, $G-B$ and $G-A-B$.

Introduction. In the present paper we shall consider simple graphs without loops and multiple edges, and three polynomials associated with them. These are the characteristic [2], the matching [1,3] and the $\mu$-polynomial [5]. They will be denoted by $\varphi(G)$, a $\alpha(G)$ and $\mu(G)$, respectively with $G$ standing for the corresponding graph.

Let $G$ be a graph with $n$ vertices, $v_1, v_2, \ldots, v_n$. Its adjacency matrix $A$ is square matrix of order $n$ whose element in the $i$-th row and $j$-th column is equal to one if the vertices $v_i$ and $v_j$ are adjacent, and is equal to zero otherwise. The characteristic polynomial of $A$ is called the characteristic polynomial of the respective graph [2]. Hence, if $I$ is the unit matrix of order $n$ then $\varphi(G) = \varphi(G, x) = \det(xI - \text{adj}(A))$.

Denoting by $m(G, k)$ the number of selections of $k$ independent edges of the graph $G$ (i. e. the number of its $k$-matchings), the matching polynomial of $G$ is defined as [1,3]

$$\alpha(G) = \alpha(G, x) = x^n + \sum_{k=1}^{n/2} (-1)^k m(G, k) x^{n-2k}.$$ 

If the graph $G$ is acyclic, then by definition, $\mu(G) = \alpha(G)$. Since the characteristic and the matching polynomial of an acyclic graph coincide $[1,3,4]$, in this case we also have $\mu(G) = \varphi(G)$.

In order to define the $\mu$-polynomial of a cyclic graph, suppose that $G$ possesses $r$ ($r > 0$) circuits $Z_1, \ldots, Z_r$, and associate a parameter $t_i$ with $Z_i, i = 1, \ldots, r$. 

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Then [5]

\[
\mu(G) = \alpha(G) + 2 \sum_i t_i \alpha(G - Z_i) + 4 \sum_{i<j} t_i t_j \alpha(G - Z_i - Z_j) \\
- \cdots + (-2)^r t_1 t_2 \cdots t_r \alpha(G - Z_1 - Z_2 - \cdots - Z_r)
\]

(1)

with the following conventions:

(a) If among the circuits \(Z_{i_1}, Z_{i_2}, \ldots, Z_{i_k}\) of \(G\) at least two of them possess at least one common vertex, then \(\mu(G - Z_{i_1} - Z_{i_2} - \cdots - Z_{i_k}) = 0\).

(b) If the circuits \(Z_{i_1}, Z_{i_2}, \ldots, Z_{i_k}\) embrace all the vertices of \(G\), then \(\mu(G - Z_{i_1} - Z_{i_2} - \cdots - Z_{i_k}) = 1\).

The \(\mu\)-polynomial is a generalization of both the matching and the characteristic polynomial. From (1) it is evident that for \(t_1 = t_2 = \cdots = t_r = 0\), \(\mu(G)\) reduces to a \(\alpha(G)\). It can be shown [5] that for \(t_1 = t_2 = \cdots = t_r = 1\), \(\mu(G)\) coincides with \(\varphi(G)\).

The concept of the \(\mu\)-polynomial was developed in connection with some problems of theoretical chemistry. The chemical applications of the \(\mu\)-polynomial are elaborated in [5], where a number of its basic properties has also been established. Among them we shall need the following three.

If the graph \(G\) is composed of components \(G_1, G_2, \ldots, G_c\), then we shall write \(G = G_1 + G_2 + \cdots + G_c\).

**Lemma 1.** \(\mu(G_1 + GZ + \cdots + G_c) = \mu(G_1) \mu(G_2) \cdots \mu(G_c)\).

**Lemma 2.** Let \(G\) be an arbitrary graph and \(u\) its vertex. Then

\[
\mu(G) = x \mu(G - u) - \sum_j \mu(G - u - v_j) - 2 \sum_k t_k \mu(G - Z_k).
\]

(2)

The first summation on the r. h. s. of (2) goes over all vertices \(v_j\) which are adjacent to \(u\); the second summation goes over all circuits \(Z_k\) which contain the vertex \(u\).

**Lemma 3.** Let \(e\) be an edge of \(G\), connecting the vertices \(u\) and \(v\). If \(e\) does not belong to any circuit of \(G\), then \(\mu(G) = \mu(G - e) - \mu(G - u - v)\).

For the characteristic and matching polynomial of a graph and some of its subgraphs two peculiar relations hold.

**Lemma 4.** If \(G\) is a graph and \(u\) and \(v\) are two distinct vertices, of \(G\) then

\[
\varphi(G - u) \varphi(G - v) - \varphi(G) \varphi(G - u - v) = \left[ \sum_{i} \varphi(G - P_i) \right]^2
\]

(3)

\[
\alpha(G - u) \alpha(G - v) \alpha(G - u - v) = \sum_{i} [\alpha(G - P_i)]^2
\]

(4)
In both expressions $P_1$ denotes a path and the summations go over all paths in $G$, which connect the vertices $u$ and $v$.

Formula (3) is a graph-theoretical reinterpretation of a long-known result for determinants [7], whereas (4) was discovered by Hellmann and Lieb [6].

As a matter of fact, in the theory of determinants the following result of Jacobi from 1833 is known [7, Theorem 1.5.3]. Let

$$D = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

be a determinant of order $n$. Let $M$ be a $k$-rowed minor of $D$, $M^*$ the corresponding minor of the adjugate of $D$ and $\tilde{M}$ the cofactor of $M$ in $D$. Then $M^* = D^{k-1}\tilde{M}$. For $k = 2$ we get as a special case of the above equation

$$\begin{vmatrix} A_{uu} & A_{uv} \\ A_{vu} & A_{vv} \end{vmatrix} = D \cdot D_{uv,uv}$$

where $A_{uv}$ is the cofactor of the element $a_{uv}$ and $D_{uv,uv}$ is the determinant of order $n-2$ obtained when the $r$-th and the $s$-th rows and columns are deleted from $D$. This yields $A_{uu}A_{vv} - D \cdot D_{uv,uv} = (A_{uv})^2$.

Suppose now that $D$ is equal to $\det(xI - A)$. Then from the definition of the characteristic polynomial of a graph, we immediately have $D = \phi(G)$, $A_{uu} = \phi(G - u)$, $A_{vv} = \phi(G - v)$ and $D_{uv,uv} = \phi(G - u - v)$. The fact that

$$A_{uv} = \sum_i \phi(G, P_i)$$

is just another formulation on Coates' formula [2, p. 47].

**The main results.** In this section we report some relations for the $\mu$-polynomial, whose form is similar to that of eqs. (3) and (4). The following two theorems and their corollaries are our main results.

Let $A$, $B$, $X$ and $Y$ be rooted graphs. Let $H$ be another graph and $u$ and $v$ two distinct vertices of $H$. Construct the graph $G$ by identifying the roots of $A$ and $X$ with $u$, and by identifying the roots of $B$ and $Y$ with $v$ (Fig. 1).
Theorem 1. Let $A^i$, $B^j$, $X^i$ and $Y^j$ denote the subgraphs obtained by deleting the rooted vertex from $A$, $B$, $X$ and $Y$, respectively. Then,

$$
\mu(G - A)\mu(G - B) - \varphi(G)\varphi(G - A - B) = \mu(A^i)\mu(B^j)\mu(X^i)\mu(Y^j) - [\mu(H - u)\mu(H - v) - \mu(H)\mu(H - u - v)].
$$

(5)

Corollary 1.1.

$$
\varphi(G - A)\varphi(G - B) - \varphi(G)\varphi(G - A - B) = [\varphi(A^i)\varphi(B^j)]^{-1} \left[ \sum_i \varphi(G - R_i) \right]^2.
$$

Corollary 1.2.

$$
\alpha(G - A)\alpha(G - B) - \alpha(G)\alpha(G - A - B) = [\alpha(A^i)\alpha(B^j)]^{-1} \sum_i \alpha(G - P_i)\alpha(G - B - P_i).
$$

Corollary 1.3.

The summations in Corollaries 1.1–1.3 go over all paths $P_i$ of the graph $G$, connecting the vertices $u$ and $v$.

Theorem 2. Let $H$ be a graph and $u$ and $v$ two distinct vertices of $H$. If $u$ and $v$ are connected by a unique path $P$, then

$$
\mu(H - u)\mu(H - v) - \mu(H)\mu(H - u - v) = [\mu(H - P)]^2.
$$

(6)

Corollary 2.1. If the vertices $u$ and $v$ of the graph $G$ (from Theorem 1) are connected by a unique path $P$, then

$$
\mu(G - A)\mu(G - B) - \mu(G)\mu(G - A - B) = \mu(G - A - P)\mu(G - B - P).
$$

Proof. In order to prove Theorem 1 we need an auxiliary result.

Lemma 5. Let $R_1, R_2, \ldots, R_m$ be rooted graphs and $u_1, u_2, \ldots, u_m$, the corresponding roots. Construct the graph $R$ by identifying the roots of all $R_i$, $i = 1, 2, \ldots, m$. The vertex so obtained will be denoted by $u$. Then

$$
\mu(R) = \mu(R_1)\mu(R_2)\cdots\mu(R_m) + \mu(R'_1)\mu(R_2)\cdots\mu(R'_m) + 
\cdots + 
\mu(R'_1)\mu(R_2)\cdots\mu(R'_m) - (m - 1)\mu(R'_1)\mu(R'_2)\cdots\mu(R'_m)
$$

(7)

where $R'_i = R_i - u_i, i = 1, 2, \ldots, m$.

Proof. Since the vertex $u$ is a cutpoint in $R$, it cannot happen that a circuit of $R$ lies partially in $R_i$ and partially $R_j$, $i \neq j$. Then applying Lemma 2 we get

$$
\mu(R) = x\mu(R - u) - \sum_{i=1}^m \sum_{j \neq i} \mu(R - u - v_{j_i}) - 2 \sum_{i=1}^m \sum_{k_i} t_{k_i}\mu(R - Z_{k_i})
$$

(8)

where $v_{j_i}$ denotes a vertex of $R_i$ which is adjacent to $u_i$ and $Z_{k_i}$ denotes a circuit of $R_i$ which contains the vertex $u_i$; the appropriate summations go over all vertices $v_{j_i}$ and all circuits $Z_{k_i}$, respectively.
From the construction of the graph $R$ it is evident that
\[
R - u = R'_1 + R'_2 + \cdots + R'_m \tag{8}
\]
and bearing in mind Lemma 1 we transform (8) into
\[
\mu(R) = x \prod_{h=1}^{m} \mu(Rh') - \sum_{i=1}^{m} \mu(Rh') \left[ \sum_{j_i} \mu(R_i - u_i - v_{j_i}) + 2 \sum_{k_i} t_{k_i} \mu(R_i - Z_{k_i}) \right] \tag{9}
\]
On the other hand, application of Lemma 2 to $R_i$ gives
\[
\mu(R_i) = x \mu(R_i - u_i) - \sum_{j_i} \mu(R_i - u_i - v_{j_i}) - 2 \sum_{k_i} t_{k_i} \mu(R_i - Z_{k_i})
\]
which combined with (9) gives
\[
\mu(R) = x \prod_{h=1}^{m} (Rh') + \sum_{i=1}^{m} \mu(Rh') [\mu(R_i) - x \mu(R_i')].
\]
Formula (7) follows now immediately. \( \square \)

Proof of Theorem 1. Lemma 5 can, of course, be used for all graphs possessing cutpoints. Since the vertices $u$ and $v$ of the graph $G$ are cutpoints (see Fig. 1) we may apply formula (7) to $G$ and its subgraphs $G - A$ and $G - B$.

![Fig. 2](image)

Defining the graph $G_1$ as obtained by identifying the roots of $B$ and $Y$ with the vertex $v$ of $H$ (see Fig. 2), we arrive at the following special case of (7):
\[
\mu(C) = \mu(A) \mu(X') \mu(G_1 - u) + \mu(A') \mu(X) \mu(G_1 - u) + \mu(A') \mu(X') \mu(G_1) - 2x \mu(A') \mu(X') \mu(G_1 - u). \tag{10}
\]
Let the graph $G_2$ be obtained, in analogy to $G_1$, by identifying the roots of $A$ and $X$ with the vertex $u$ of $H$ (see Fig. 2). Then we immediately conclude that

\[ G - A = X' + (G_1 - u), \quad G - B = Y' + (G_2 - v) \quad \text{and} \quad G - A - B = X' + Y' + (H - u - v) \]

and therefore

\[ \mu(G - A) = \mu(X')\mu(G_1 - u), \quad \mu(G - B) = \mu(Y')\mu(G_2 - v) \]
\[ \mu(G - A - B) = \mu(X')\mu(Y')\mu(H - u - v). \]

On the other hand, by Lemma 5,

\[ \mu(G_1) = \mu(B)\mu(Y')\mu(H - v) + \mu(B')\mu(Y)\mu(H - v) + \mu(B')\mu(Y')\mu(H - v) - 2x\mu(B')\mu(Y')\mu(H - v) \]
\[ \mu(G_1 - u) = \mu(B)\mu(Y')\mu(H - u - v) + \mu(B')\mu(Y)\mu(H - u - v) + \mu(B')\mu(Y')\mu(H - u - v) - 2x\mu(B')\mu(Y')\mu(H - u - v) \]
\[ \mu(G_2 - v) = \mu(A)\mu(X')\mu(H - u - v) + \mu(A')\mu(X)\mu(H - u - v) + \mu(A')\mu(X')\mu(H - v) - 2x\mu(A')\mu(X')\mu(H - u - v). \]

Substituting eqs. (10)–(13) into the l. h. s. of formula (5) we obtain its r. h. s. after a lengthy calculation.

Corollary 1.1 follows for $t_1 = t_2 = \cdots = t_r = 1$, by taking into account eq. (3) and the fact that $G - P_t = A^r + B^r + X + Y + (H - P_1)$. Corollary 1.2 is obtained in a similar manner for $t_1 = t_2 = \cdots = t_r = 0$ using eq. (4). Corollary 1.3 is based on the fact that because of $(G - A - P_t) + (G - B - P_t) = A^r + B^r + X + Y + (H - P_1) + (H - P_t)$, we have

\[ \alpha(A')\alpha(B')\alpha(X')\alpha(Y')\alpha(H - P_1)^2 = \alpha(G - A - P_t)\alpha(G - B - P_t). \]

**Proof of Theorem 2** will be performed by induction on the length $p$ of the path $P$.

Let $H_0, H_1, \ldots, H_p$ be rooted graphs whose roots are denoted by $v_0, v_1, \ldots, v_p$, respectively. Then the graph $H$ (from Theorem 2) can be constructed by joining the vertices $v_{i-1}$ and $v_i$ by a new edge $e_i$, $i = 1, \ldots, p$ (see Fig. 3). In this notation, $v_0 \equiv u$ and $v_p \equiv v$.

One should observe that the edges $e_i$ cannot belong to circuits, and thus Lemma 3 is applicable to them.
For $p = 0$, eq. (6) is fulfilled in a trivial manner since then $u \equiv v$ and, by definition, $\mu(H - u - v) \equiv 0$.

For $p = 1$, Lemma 3 gives $\mu(H) = \mu(H_0)\mu(H_1) - \mu(H_0 - u)\mu(H_1 - v)$ and since

$$
\mu(H - u) = \mu(H_0 - u)\mu(H_1), \quad \mu(H - v) = \mu(H_0)\mu(H_1 - v), \\
\mu(H - u - v) = \mu(H - P)\mu(H_0 - u)\mu(H_1 - v)
$$

one immediately verifies that (6) holds.

Suppose now that $p > 1$ and that (6) holds for the graph $H'$ and its vertices $v_1$ and $v_{p-1}$ (see Fig. 4). For convenience we shall write $v_1 = u', v_{p-1} = v'$. Applying Lemma 3 to the edges $e_1$ and $e_p$ of $H$ and using Lemma 1, we arrive at

$$
\mu(H) = \mu(H_0)\mu(H_p)\mu(H') - \mu(H_0 - v_0)\mu(H_p)\mu(H' - u'), \\
-\mu(H_0)\mu(H_p - v_p)\mu(H' - v') + \mu(H_0 - v_0)\mu(H_p - v_p)\mu(H' - u'v').
$$

In addition to this,

$$
\mu(H - u) = \mu(H_0 - v_0)[\mu(H_p)\mu(H') - \mu(H_p - v_p)\mu(H' - u')], \\
\mu(H - v) = \mu(H_p - v_p)[\mu(H_0)\mu(H') - \mu(H_0 - v_0)\mu(H' - u')], \\
\mu(H - u - v) = (H_0 - v_0)\mu(H_p - v_p)\mu(H').
$$

Substituting all these relations into the l. h. s. of eq. (6) one obtains

$$
\mu(H - u)\mu(H - v) - \mu(H)\mu(H - u - v) = \\
\mu(H_0 - v_0)^2\mu(H_p - v_p)^2[\mu(H' - u')\mu(H' - v') - \mu(H')\mu(H' - u' - v')].
$$

According to the induction hypothesis,

$$
\mu(H' - u')\mu(H' - v') - \mu(H')\mu(H' - u' - v') - [\mu(H' - P)]^2
$$

where $P$ is the (unique) path connecting $v_1$ and $v_{p-1}$ in $H'$. Bearing in mind that $H' - P = (H_1 - v_1) + (H_2 - v_2) + \cdots + (H_{p-1} - v_{p-1})$ we conclude that

$$
\mu(H - u)\mu(H - v) - \mu(H)\mu(H - u - v) = \\
= [\mu(H_0 - v_0)\mu(H_1 - v_1)\mu(H_2 - v_2)\cdots\mu(H_p - v_p)]^2
$$

which is equivalent to eq. (6). This proves Theorem 2. □

**Discussion.** It see that Theorems 1 and 2 are special cases of a more general result, which, however remains still to be discovered. We conjecture the following relation for the matching polynomial.

Let $G$ be a graph and $A$ and $B$ its two subgraphs, such that $A$ and $B$ have disjoint vertex sets. Let $P_1, P_2, \ldots, P_s$ be the paths in $G$ whose one endpoint
belongs to $A$, the other endpoint to $B$, and no other vertex belongs to either $A$ or $B$. then

$$
\alpha(G - A)\alpha(G - B) - \alpha(G)\alpha(G - A - B) = \sum_{i} \alpha(G - A - P_i)\alpha(G - B - P_i) - \\
- \sum_{i < j} \alpha(G - A_i - P_i - P_j)\alpha(G - B - P_i - P_j) + \cdots + \\
+(-1)^{n-1}\alpha(G - A - P_1 - P_2 - \cdots - P_n)\alpha(G - B - P_1 - P_2 - \cdots - P_n)
$$

(15)

where the convention is that whenever at least two among the paths $P_i, P_{i+1}, \ldots, P_n$ have at least one common vertex, then $\alpha(G - A - P_1 - P_2 - \cdots - P_n) \equiv \alpha(G - B - P_1 - P_2 - \cdots - P_n) \equiv 0$.

If both $A$ and $B$ are one-vertex graphs, then (15) reduces to (4). Another special case of eq. (15), namely when only $B$ is a one-vertex graph, reads

$$
\alpha(G - A)\alpha(G - v) - \alpha(G)\alpha(G - A - v) = \sum_{i} \alpha(G - A - P_i)\alpha(G - v - P_i)
$$

and has been established previously [6]. Corollary 1.3 is a third special case of the formula (15).

REFERENCES