A NOTE ON A BERMOND’S CONJECTURE

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Abstract. If $n \geq 2$ is prime and $k \leq n$, then the arcs of $K_n^*$ can be partitioned into $k$-cycles iff $n(n-1) \equiv 0 \pmod{k}$.

Let $n$ and $k$ be two non-negative integers. We denote by $K_n^*$ the complete symmetric digraph (directed graph) on $n$ vertices, having $n(n-1)$ arcs, i.e., every ordered pair of vertices is joined by exactly one arc. By a $k$-cycle, we mean an elementary cycle (directed cycle) of length $k$. A packing is a set of arc-disjoint cycles of the digraph. A covering is a set of cycles covering all the arcs of a digraph. If a digraph has a packing which is also a covering, we say that the arcs of the digraph can be partitioned into cycles.

From [1] he have the conjecture 4.3.1 due to Bermond: “The arcs of $K_n^*$ can be partitioned into $k$-cycles iff $n(n-1) \equiv 0 \pmod{k}$”. If $n \geq 2$ is prime and $k \leq n$, then the conjecture is true, i.e., we have the following theorem.

Theorem. If $n \geq 2$ is prime and $k \leq n$, then the following are equivalent:
(a) The arcs of $K_n^*$ can be partitioned into $k$-cycles; (b) $n(n-1) \equiv 0 \pmod{k}$.

Proof. The case $n = 2$ is trivial. So, let us suppose $n \geq 3$. Obviously, (a) implies (b), since a necessary condition for the existence of a partition into $k$-cycles of $K_n^*$ is that the number $n(n-1)$ to be divisible by $k$. Now, we prove the converse. Because $n \geq 3$ and $n$ is prime, then $n$ is odd. If $k = n$, the theorem follows by [1, Theorem 4.1.4]. So, let $k < n$. Then, according to (b), $k$ divides $n-1$ since $n$ is prime. Let $F$ be a finite field with $n$ elements (e.g., $GF(n)$), and $1_F$ the multiplicative identity of $F$. Since the multiplicative group of $F$ contains $n-1$ elements and $k$ divides $n-1$, then there exists $g \in F$ of order $k$, i.e., $g^k = 1_F$ and $g \neq 1_F$. We shall identify the vertices of $K_n^*$ with the elements of $F$ and, for an arbitrary arc $(x, y)$ of $K_n^*$, we define the following sequence of vertices:

$$x_i = x + (y - x)(g^i - 1_F)/(g - 1_F), \quad i = 1, 2, \ldots, k. \quad (1)$$

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Obviously, \( g^i = g^j \) iff \( i \equiv j \pmod{k} \). Therefore, the sequence
\[
C(k, x, y) = (x, y = x_1, x_2, \ldots, x_{k-1}, x_k = x)
\]
is a \( k \)-cycle of \( K_n^* \). Let \( C(k, x', y') = (x', y' = x'_1, x'_2, \ldots, x'_{k-1}, x'_k = x') \) be another \( k \)-cycle of \( K_n^* \), obtained according to (1), such that \( C(k, x, y) \) and \( C(k, x, y') \) have an arc in common, i.e., \((x_i, x_{i+1}) = (x'_j, x'_{j+1})\). It follows that
\[
x_i = x'_j \quad \text{and} \quad x_{i+1} = x'_{j+1}.
\] (2)
From (1), by straightforward calculus, we obtain
\[
x_{i+2} - x_{i+1} = g(x_{i+1} - x_i),
\] (3)
\[
x'_{j+2} - x'_{j+1} = g(x'_{j+1} - x'_j).
\] (4)
Thus, from (2) - (4), we have \( x_{i+2} = x'_{j+2} \), and, by recurrence, we obtain \( x_{i+t} = x'_{j+t} \), \( 0 \leq t \leq k - 1 \) (the indices are taken modulo \( k \)). Hence,
\[
C(k, x, y) = C(k, x', y').
\] (5)
Considering the set of \( k \)-cycles obtained according to (1) for all the possible choices of the arc \((x, y)\), and having in view (5), we obtain a partition of \( K_n^* \), by taking all the distinct \( k \)-cycles. Thus, the theorem is completely proved.

REFERENCES