CONDITIONAL PROBABILITY IN NONSTANDARD ANALYSIS

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Abstract. In this paper we apply the theory of Loeb measure to conditional probability for hyperfinite Loeb spaces. We show that conditional probability \( \sim P(\cdot | A) \) on a Loeb space \( (V, \mathcal{M}(\sim P), \sim P) \) for \( A \in \mathcal{P}(V) \) \( (P(A) > 0 \) and \( P(A) \approx 0 \) \( \) is a Loeb measure and for \( A \in \mathcal{M}(\sim P) \) \( \sim P(A) > 0 \) can be represented by a Loeb measure. For the case \( A \in \mathcal{M}(\sim P) \) we prove that there exists a set \( C \in \mathcal{P}(V) \) such that \( \sim P(\cdot | A) \) is equal to the Loeb conditional probability \( L(P(\cdot | C)) \). We introduce internal conditional probability relative to an internal subalgebra \( \mathfrak{A} \) of \( \mathcal{P}(V) \) as in case of finite standard probability spaces. We show, analogously to a well-known probability result, that internal conditional probability \( P(A/\mathfrak{A}), A \in \mathcal{P}(V) \), and internal conditional expectation \( E(X/\mathfrak{A}) \), \( X \) is \( S \)-integrable, are \( P \)-a. s. unique, in nonstandard sense, random variables on \( (V, \mathfrak{A}, P) \). Finally, we give a nonstandard characterization of conditional probability \( \sim P(A/M(\mathfrak{A})) \), \( A \in \mathcal{M}(\sim P) \) on a Loeb space \( (V, \mathcal{M}(\sim P), \sim P) \). We prove that there exists a set \( C \in \mathcal{P}(V) \) such that \( P(C/\mathfrak{A}) \) is the lifting of \( \sim P(A/M(\mathfrak{A})) \).

Introduction. In this paper we concern ourselves with conditional probability for hyperfinite Loeb spaces. We use the well-known results from the theory of Loeb measure [8] and nonstandard probability [2], [10] and the methodology developed by P. Loeb, J. Keisler, R. Anderson and others.

In the first part we define internal conditional probability \( P(\cdot | A), A \in \mathcal{P}(V) \) for a hyperfinite probability space \( (V, \mathcal{P}(V), P) \) and give the nonstandard representation of conditional probability \( \sim P(\cdot | A), A \in \mathcal{M}(\sim P) \) on the Loeb space \( (V, \mathcal{M}(\sim P), \sim P) \). We show that for \( A \in \mathcal{P}(V) \) with \( P(A) > 0 \) and \( P(A) \approx 0 \) \( \) is a Loeb measure on \( (V, \mathcal{M}(\sim P)) \) and for \( A \in \mathcal{M}(\sim P) \) with \( \sim P(A) > 0 \) there exists a set \( C \in \mathcal{P}(V) \) such that \( \sim P(\cdot | A) \) can be represented by the Loeb conditional probability \( L(P(\cdot | C)) \).

In the second part we define, analogously to the definition of internal conditional expectation \( E(X/\mathfrak{A}), [10] \), internal conditional probability \( P(A/\mathfrak{A}) \), \( A \in \mathcal{P}(V) \), is an internal subalgebra of \( \mathcal{P}(V) \) for a hyperfinite probability space \( (V, \mathcal{P}(V), P) \). We show that so-introduced \( P(A/\mathfrak{A}) \) \( (E(X/\mathfrak{A}) \) as well) is
\[ P - a. s. \text{ unique random variable on } (V, \mathfrak{A}, P). \] The \( P - a. s. \text{ uniqueness of the } P(A/\mathfrak{A}) \) \( E(X/\mathfrak{A}) \) is introduced in theorem 4, corresponding to the same concept from standard probability. Finally, we give a nonstandard characterization of conditional probability \( P(A/\mathfrak{M}(\mathfrak{A})) \) \( (A \in \mathfrak{M}(\sim P), (A \in \mathfrak{M}(\sim P) \mathfrak{M}(\mathfrak{A}) \text{ is a sub-\( \sigma \)-algebra of } \mathfrak{M}(\sim P)) \) on a Loeb space \( (V, \mathfrak{M}(\sim P), \sim P) \). We show that for \( A \in \mathfrak{M}(\sim P) \) there exists a set \( B \in^{*} \mathfrak{P}(V) \) such that \( P(B/\mathfrak{A}) \) is lifting of \( \sim P(A/\mathfrak{M}(\mathfrak{A})) \).

We assume \( (V, \mathfrak{P}(V), P) \) to be a hyperfinite probability space and \( (V, \mathfrak{M}(\sim P), \sim P) \) a Loeb space constructed from it, i.e. Loeb measure \( \sim P \) is defined by

\[
\sim P(F) = \inf \{s(P(A)) \mid F \subseteq A \text{ and } A \in^{*} \mathfrak{P}(V)\} \\
= \sup \{s(P(A)) \mid A \subseteq F \text{ and } A \in^{*} \mathfrak{P}(V)\}
\]

for \( F \subseteq V \) and \( \mathfrak{M}(\sim P) \) is a \( \sigma \)-algebra of all \( \sim P \)-measurable sets \( F \subseteq V \).

According to standard probability, for \( A \in^{*} \mathfrak{P}(V) \) with \( P(A) > 0 \) we define internal conditional probability \( P(\cdot/\mathfrak{A}) \) of an internal event \( B \) relative to \( A \) by

\[
P(B/A) = P(A \cap B)/P(A)
\]

It is easy to show that \( (V, B(V), P(\cdot/\mathfrak{A})) \) is a hyperfinite probability space, so it gives rise to a Loeb space denoted by \( (V, \mathfrak{M}(L(P(\cdot/\mathfrak{A}))), L(P(\cdot/\mathfrak{A}))) \).

On the other hand, for a Loeb space \( (V, \mathfrak{M}(\sim P), \sim P) \), conditional probability \( \sim P(\cdot/\mathfrak{A}) \) of an event \( B \in \mathfrak{M}(\sim P) \) relative to \( A \in \mathfrak{M}(\sim P), \sim P(A) > 0 \), is standardly defined by

\[
\sim P(B/A) = \sim P(A \cap B)/\sim P(A)
\]

It is well known that \( \sim P(\cdot/\mathfrak{A}) \) is a probability measure on \( (V, \mathfrak{M}(\sim P)) \) but not necessary a Loeb measure. However, for \( A \in^{*} \mathfrak{P}(V) \) with \( P(A) > 0 \) \( P(A) \approx 0 \), \( \sim P(\cdot/\mathfrak{A}) \) is a Loeb measure and we shall show it in this paper.

Let \( \sigma(\sim P(\cdot/\mathfrak{A})) \) be \( \sigma \)-algebra of all \( \sim P(\cdot/\mathfrak{A}) \)-measurable sets, i.e.

\[
\sigma(\sim P(\cdot/\mathfrak{A})) = \{F \subseteq V \mid F \cap A \in \mathfrak{M}(\sim P)\}
\]

It is obvious that \( \mathfrak{M}(\sim P) \subseteq \sigma(\sim P(\cdot/\mathfrak{A})) \). From the theory of Loeb measure we know that \( \sim P \) is a complete measure on \( (V, \mathfrak{M}(\sim P)) \) and \( \mathfrak{M}(\sim P) \) is a completion of \( L(V) \) relative to \( \sim P \). We have the same for \( \sim P(\cdot/\mathfrak{A}) \):

**Lemma 1.** Probability measure \( \sim P(\cdot/\mathfrak{A}) \) is a complete measure on \( (V, \mathfrak{M}(\sim P)) \) and \( \mathfrak{M}(\sim P) \) is a completion of \( L(V) \) relative to \( \sim P(\cdot/\mathfrak{A}) \) in the sense that for \( F \in \mathfrak{M}(\sim P) \) there exist sets \( Z \subseteq L(V) \) and \( U \subseteq V \) such that

\[
F = Z \cup N, \quad N \subseteq U, \quad \text{and} \quad \sim P(U/A) = 0.
\]

**Proof** Let \( F \in \mathfrak{M}(\sim P) \), \( \sim P(F/A) = 0 \) and \( M \subseteq F \). Then is \( F \cap A \in \mathfrak{M}(\sim P) \), \( M \cap A \subseteq F \cap A \) and \( \sim P(F \cap A) = 0 \). Since \( \sim P \) is a complete measure, \( M \cap A \in \mathfrak{M}(\sim P) \) and \( \sim P(M \cap A) = 0 \). This implies that \( M \) is \( \sim P(\cdot/\mathfrak{A}) - \) measurable and \( \sim P(M/A) = 0 \).
Let $F \in \mathcal{M}(\neg P)$. Then there exist sets $Z \in \mathcal{L}(V)$ and $N \subseteq V$ such that $F = Z \cup N$, $N \subseteq U$ and $\neg P(U) = 0$. Since $U \cap A \subseteq U$ and $\neg P$ is a complete measure, it follows that $\neg P(U \cap A) = 0$, i.e. $\neg P(U/A) = 0$. Hence $\mathcal{M}(\neg P)$ is a completion of $\mathcal{L}(V)$ relative to $\neg P(A)$.

The next theorem shows that conditional probability $\neg P(\cdot | A)$ for $A \in^* \mathcal{P}(V)$, $P(A) > 0$, and $P(A) \approx 0$ is a Loeb measure on $(V, \mathcal{M}(\neg P))$.

**Theorem 1.** Let $A \in^* \mathcal{P}(V)$, $P(A) > 0$ and $P(A) \approx 0$. Then $(V, \mathcal{M}(\neg P), \neg P(\cdot | A))$ is a Loeb probability space.

**Proof.** We show that $\neg P(\cdot | A)$ is a Loeb measure obtained from internal conditional probability $P(\cdot | A)$. Using notations already defined we prove that

$$L(P(B/A)) = \neg P(B/A) \quad \text{for } B \in \mathcal{M}(\neg P) \tag{1}$$

Let $F \in \mathcal{L}(V)$. The definition of Loeb measure

$$L(P(F/A)) = \sup\{\text{st}(P(C/A)) | C \in^* \mathcal{P}(V), C \subseteq F\} = \inf\{\text{st}(P(D/A)) | (D \in^* \mathcal{P}(V), D \supseteq F)$$

implies that for $\varepsilon \in^* \mathbb{R}^+$ there exist sets $C, D \in^* \mathcal{P}(V)$ such that

$$\neg P(C/A) \leq \neg P(F/A) \neg P(D/A) \quad \text{and} \tag{2}$$

$$\neg P(D/A) - \varepsilon < L(P(F/A)) < \neg P(C/A) + \varepsilon \tag{3}$$

Relations (2) and (3) imply

$$\neg P(F/A) - \varepsilon < L(P(F/A)) < \neg P(F/A) + \varepsilon \quad \text{i.e.} \quad L(P(F/A)) = \neg P(F/A)$$

Let $F \in \mathcal{M}(\neg P)$. Then, according to [8], there exist sets $C, D \in \mathcal{L}(V)$ such that $C \subseteq F \subseteq D$ and $\neg P(C) = \neg P(F) = \neg P(D)$. We show that $F \in \mathcal{M}(L(P(\cdot | /A)))$ and that $L(P(F/A)) = \neg P(F/A)$. Since

$$\neg P(DUA) = \neg P(D) + \neg P(A) - \neg P(D \cap A) \leq \neg P(D) + \neg P(A) - \neg P(C \cap A) = \neg P(C) + \neg P(A) - \neg P(C \cap A) = \neg P(C \cup A) \leq \neg P(D \cup A)$$

it follows that $\neg P(D \cap A) = \neg P(C \cap A)$, whence, and from $\neg P(C \cap A) \leq \neg P(F \cap A) \leq \neg P(D \cap A)$ we get

$$\neg P(C/A) = \neg P(F/A) = \neg P(D/A) \tag{4}$$

From (1) and (4) it follows that

$$L(P(C/A)) = L(P(D/A)) = \neg P(F/A) \tag{5}$$
and \( L(P(DC/A)) = 0 \). For the set \( F \in \mathcal{M}(\neg P) \) we have the following representation:

\[
F = C \cup (FC) \text{ where } C \in \mathcal{I}(V), FC \subseteq DC \text{ and } L(P(DC/A)) = 0.
\]

So \( F \in \mathcal{M}(L(P(:/A))) \). Since \( C \subseteq F \subseteq D \text{ and } C,D,F \in \mathcal{M}(L(P(:/A))) \) we have that \( L(P(C/A)) \leq L(P(F/A)) \leq L(P(D/A)) \) which, in view of (5), implies

\[
\neg P(F/A) \leq L(P(F/A)) \leq \neg P(F/A) \quad \text{i.e.} \quad L(P(F/A)) = \neg P(F/A)
\]

Later on, whenever \( A \in ^* \mathfrak{P}(V) \), \( P(A) > 0 \) and \( P(A) \approx 0 \), the conditional probability \( \neg P(A) \) on \((V, \mathcal{M}(\neg P))\) will be denoted by \( L(P(:/A)) \); assuming that it is a Loeb measure.

We now prove a representation theorem for conditional probability \( \neg P(:/A) \) \((A \in \mathcal{M}(\neg P) \text{ and } \neg P(A) > 0 \) on \((V, \mathcal{M}(\neg P))\). We shall show that there exists a set \( C \in ^* \mathfrak{P}(V) \) with \( P(C) > 0 \) and \( P(C) \approx 0 \) such that the conditional probability \( \neg P(:/A) \) is equal to the Loeb conditional probability \( L(P(:/C)) \).

**Theorem 2.** Let \( A \in \mathcal{M}(\neg P) \) with \( \neg P(A) > 0 \). Then, there exists a set \( C \in ^* \mathfrak{P}(V) \) with \( P(C) > 0 \) and \( P(C) \approx 0 \) such that

\[
L(P(F/C)) = \neg P(F/A), \quad \text{for any } F \in \mathcal{M}(\neg P)
\]

**Proof.** According to [8] exists a set \( C \in ^* \mathfrak{P}(V) \) such that \( \neg P(C \Delta A) = 0 \). We show that \( P(C) > 0 \) and \( P(C) \approx 0 \); For sets \( A,C \subseteq V \) we have that \( C \setminus A \subseteq C \Delta A \), and \( A \setminus C \subseteq C \Delta A \), so, by completeness of measure \( \neg P \)

\[
\neg P(C \setminus A) = \neg P(A \setminus C) = 0
\]

Since \( C = (C \setminus A) \cup (C \cap A) \) and \( A = (A \setminus C) \cup (A \cap C) \) and sets \( A,C \) satisfy (1)

\[
\neg P(C) = \neg P(C \setminus A) + \neg P(C \cap A) = \neg P(C \cap A) = \neg P(A \setminus C) + (C \cap A) = \neg P(A)
\]

Hence \( P(C) > 0 \) and \( P(C) \approx 0 \),

Let \( F \in \mathcal{M}(\neg P) \). Then

\[
F \cap C = (F \cap A \cap C) \cup ((C \setminus A) \cap F) \text{ and } F \cap A = (F \cap A \cap C) \cup ((A \setminus C) \cap F) \quad (3)
\]

From (3), \( (C \setminus A) \cap F \subseteq A \Delta C \), \( (A \setminus C) \cap F \subseteq A \Delta C \), \( \neg P((C \setminus A) \cap F) = 0 \) and \( \neg P((A \setminus C) \cap F) = 0 \) it follows that

\[
\neg P((A \setminus C) = \neg P(F \cap A \cap C) + \neg P((C \setminus A) \cap F) = \neg P(F \cap A \cap C) =
\]

\[
= \neg P(F \cap A \cap C) + \neg P((A \setminus C) \cap F) = \neg P(F \setminus A)
\]

Finally, according to theorem 1 (2) and (4) imply

\[
L(P(F/C)) = \neg P(F/C) = \neg P(F \cap C) / \neg P(C) = \neg P(F \cap A) / \neg P(A) = \neg P(F/A)
\]
The following theorem is a simple consequence of the Loeb theorem [8], but can be quite useful when working in nonstandard probability.

**Theorem 3.** Let \( A \in \mathcal{M}(\sim P) \) with \( \sim P(A) > 0 \). Then, for any set \( F \in \mathcal{M}(\sim P) \) there exists a set \( C \in \mathcal{P}(V) \) such that \( \sim P(F/A) = \sim P(C/A) \).

**Proof.** For \( F \in \mathcal{M}(\sim P) \), by the Loeb theorem [8], there exists a set \( C \in \mathcal{P}(V) \) such that \( \sim P(F \triangle C) = 0 \). Since \( F \cap A = (F \cap A \cap C) \cup ((F \setminus C) \cap A) C \cap A = (F \cap A \cap C) \cup ((C \setminus F) \cap A) (F \setminus X) \cap A \subseteq F \triangle C \) and \( (C \setminus F) \cap A \subseteq F \triangle C \), by the same arguments as in theorem 2, we get that \( \sim P(F \cap A) = \sim P(C \cap A) \), i.e. \( \sim P(F/A) = \sim P(C/A) \).

In the second part of this paper we are dealing with internal conditional expectation \( E(X/\mathfrak{A}) \) of an internal random variable \( X : V \to \mathcal{P}(V) \) relative to \( \mathfrak{A} \), where \( \mathfrak{A} \) is an internal subalgebra of \( \mathcal{P}(V) \) and \( P(A/\mathfrak{A}) \) denotes the internal conditional probability of an event \( A \in \mathcal{P}(V) \) relative to \( \mathfrak{A} \).

We consider a hiperfinite probability space \( V, \mathcal{P}(V), P, A \in \mathcal{P}(V) \) and internal subalgebra of \( \mathcal{P}(V) \). The hyperfinitness of \( \mathfrak{A} \) implies, by transfer principle, that \( \mathfrak{A} \) is generated by a hiperfinite partition \( \{V_1, V_2, \ldots, V_H\} \) \( (H \in \mathcal{P}(N) \setminus N) \) of the set \( V \). It permits us to define \( P(A/\mathfrak{A}) \) in the same way as in the case of finite standard probability spaces:

\[
P(A/\mathfrak{A})(v) = \sum_{i=1}^{H} P(A/V_i)I_{V_i}(v) \quad \text{for } v \in V \tag{1}
\]

where \( P(A/V_i) = P(A \cap V_i) / P(V_i) \quad i = 1, 2, \ldots, H \). Since

\[
P(A/\mathfrak{A})(v) = \sum_{i=1}^{H} P(A/V_i)I_{V_i}(v)
\]

\[
= \sum_{i=1}^{H} (P(A \cap V_i) / P(V_i))I_{V_i}(v)
\]

\[
= \sum_{i=1}^{H} ((P(V_i))^{-1} \sum_{u \in V_i} (P(u)I_A(u), u \in V_i))I_{V_i}(v)
\]

\[
= \sum_{i=1}^{H} E(I_A/V_i)I_{V_i}(v)
\]

\[
= E(I_A/\mathfrak{A})(v) \quad \text{i.e.}
\]

\[
P(A/\mathfrak{A}) = E(I_A/\mathfrak{A}). \tag{2}
\]

in the further work we shall use both (1) and (2) as definitions of internal conditional probability.

For internal random variable \( X : V \to \mathcal{P}(V) \) on \( (V, \mathcal{P}(V), P) \) internal conditional expectation \( E(X/\mathfrak{A}) \) has already been defined [6]. In [10] it is proved...
that \( E(X/\mathfrak{A}) \) is an \( S\mathfrak{A} \)-integrable random variable on \((V, \mathfrak{A}, P)\) provided \( X \) is \( S\)-integrable. This result applied to \( P(A/\mathfrak{A}) \) implies that \( P(A/\mathfrak{A}) \) is an \( S\mathfrak{A} \)-integrable random variable on \((V, \mathfrak{A}, P)\) since \( I_A \) is \( S\)-integrable, [9]. In [10] it is proved that

\[
E(E(X/\mathfrak{A})) = E(X). \tag{i}
\]

Taking (2) as definition of \( P(A/\mathfrak{A}) \), from (i) it follows, [9], that

\[
E(P(A/\mathfrak{A})) = P(A) \tag{ii}
\]

Results (i) and (ii) make the Theorem of probability completeness for \( E(X/\mathfrak{A}) \) and \( P(A/\mathfrak{A}) \) hold for hyperfinite probability spaces.

We now prove a nonstandard version of the well known probability theorem, namely, that conditional probability and expectation relative to \( \sigma \)-subalgebra \( \mathfrak{B} \) are \( \mu \)-a. s. unique random variables on \((V, \mathfrak{B}, \mu)\), [11].

**Theorem 4.** Let \((V, * \mathfrak{B}(V), P)\) be a hyperfinite probability space, \( \mathfrak{A} \subseteq * \mathfrak{B}(V) \) an internal subalgebra generated by a hyperfinite partition \( \{V_1, V_2, \ldots, V_H\} \) (\( H \in * \mathbb{N} \)) of \( V \), \( X : VV \to * R \) an \( S \)-integrable random variable on \((V, * \mathfrak{B}(V), P)\) and \( A \in * \mathfrak{B}(V) \). Then

(i) \( \sum (X(v)P(v), v \in U) = \sum (E(X/\mathfrak{A})(v)P(v), v \in U) \) for \( U \in \mathfrak{A} \)

(ii) \( E(X/\mathfrak{A}) \) is the \( P \)-a. s. unique internal random variable on \((V, \mathfrak{A}, P)\) which satisfies (i), i.e. for any other \( S \)-integrable \( Y : V \to * R \) on \((V, \mathfrak{A}, P)\) satisfying (i)

\[
Y(v) \approx E(X/\mathfrak{A})(v) \quad \text{\( P \)-n. s.}
\]

and for any \( S \)-integrable \( H : V \to * R \) on \((V, \mathfrak{A}, P)\) with

\[
\sum (\|(H(v) - E(X/\mathfrak{A})(v))P(v), v \in V) \approx 0 \quad \text{one has}
\]

\[
\sum (H(v)P(v), v \in U) \approx \sum (E(X/\mathfrak{A})(v)P(v), v \in U) \quad \text{for } U \in \mathfrak{A}.
\]

(iii) For any set \( B \in \mathfrak{A} \)

\[
P(A \cap B) = \sum (P(A/\mathfrak{A})(v)P(v), v \in B).
\]

(iv) \( P(A/\mathfrak{A}) \) is the \( P \)-a. s. unique internal random variable on \((V, \mathfrak{A}, P)\) in the sense given in (ii).

**Proof** (i) Since \( U = \bigcup_{i=1}^{H} (U \cap V_i) \), for \( v \in U \cap V_i E(X/\mathfrak{A})(v) = E(X/U \cap V_i) \)
one has
\[
\sum_{i=1}^{H} (E(X|\mathcal{A})(v)P(v), v \in U) = \sum_{i=1}^{H} (E(X|\mathcal{A})(v)P(v), v \in (U \cap V_i))
\]
\[
= \sum_{i=1}^{H} (E(X|\mathcal{A})(v)P(v), v \in U \cap V_i))
\]
\[
= \sum_{i=1}^{H} (E(X|U \cap V_i))P(U \cap V_i)
\]
\[
= \sum_{i=1}^{H} ((P(U \cap V_i)^{-1} \sum_{u \in U \cap V_i} (X(u)P(u)), u \in U \cap V_i))P(U \cap V_i)
\]
\[
= \sum_{i=1}^{H} (X(u)P(u), u \in (U \cap V_i))
\]
\[
= \sum_{i=1}^{H} (X(u)P(u), u \in U)
\]

(ii) Let \(F(v) = E(X|\mathcal{A})(v)\). Then, in view of the Projection Theorem for Integrability, [1], \(S-\mathcal{A}\)-integrability of \(F : V \to R\) implies that \(\sim F : V \to R\) is a \(\sim P\)-integrable random variable on \((V, \mathcal{M}(\mathcal{A}), \sim P)\). If \(Y : V \to \ast R\) is any \(S-\mathcal{A}\)-integrable random variable on \((V, \mathcal{A}, P)\) which satisfies (i) then \(\sim Y : V \to R\) is a \(\sim P\)-integrable random variable on \((V, \mathcal{M}(\mathcal{A}), \sim P)\) as well. Therefore, for \(U \in \mathcal{A}\)
\[
\int_{U} \sim Y d\sim P = \text{st}(\sum_{U} (Y(u)P(u), u \in U)) = \text{st}(E(F(u)P(u), u \in U)) = \int_{U} \sim F d\sim P
\]
Let \(M \in \mathcal{M}(\mathcal{A})\). Then, by [8], there exists a set \(U \in \mathcal{A}\) such that \(P(U \Delta M) = 0)\).
Since \(U\) satisfies (1), we have
\[
\int_{M} \sim Y d\sim P = \int_{U} \sim Y d\sim P = \int_{U} \sim F d\sim P = \int_{M} \sim F d\sim P
\]
and hence \(\sim Y(v) = \sim F(v)\) P-a.s.. This implies that
\[
P\{v \in V | |Y(v) - F(v)| > n^{-1}\} \approx 0 \quad \text{for every } n \in N.
\]
According to [4, Robinson’s lemma about sequences] there exists \(h \in \ast \mathbb{N}\) such that for every \(k \in \ast \mathbb{N}\), \(k \leq h\)
\[
P\{v \in V | |Y(v) - F(v)| > k^{-1}\} \approx 0
\]
Therefore, the set \(U = \{v \in V | |Y(v) - F(v)| > h^{-}\} \approx 0\) satisfies: \(U \in \mathcal{A}, P(U) \approx 0, U \supset \{v \in V | Y(v) \neq F(v)\}\) and \(Y(v) \approx E(X|\mathcal{A})(v)\) for \(u \not\in U\). Hence
\[
Y(v) \approx E(X|\mathcal{A})(v) \quad \text{P-a.s.}
\]
Let $H : V \rightarrow * R$ be an $S\mathfrak{A}$-integrable random variable on $(V, \mathfrak{A}, P)$ with $\sum(|H(v) - F(v)|P(v), v \in V) \approx 0$. Since $\sim H$, $\sim F$ are $\sim P$-integrable random variables on $(V, \mathcal{M}(\mathfrak{A}), \sim P)$ and
\[
\int \sim H - \sim F|d\sim P = \text{st}(\sum(|H(v) - F(v)|P(v), v \in V)) = 0
\]
we have that $\sim H(v) = \sim F(v)$ $P$-a.s. Therefore, for $U \in \mathfrak{A}$
\[
\sum (H(v)P(v), v \in U) \approx \int \sim Hd\sim P = \int \sim Fd\sim P \approx \sum (F(v)P(v), v \in U) \quad \text{i.e.}
\]
\[
\sum (H(v)P(v), v \in U) \approx \sum (E(X/\mathfrak{A})(v)P(v), v \in U)
\]
(iii) According to def (2) for $P(A/\mathfrak{A})$ and (i), for $B \in \mathfrak{A}$ we have
\[
\sum (P(A/\mathfrak{A})(v)P(v), v \in B) = \sum (E(I_A/\mathfrak{A})(v)P(v), v \in B)
= \sum (I_A(v)P(v), v \in B) = P(A \cap B)
\]
(iv) Let $F(v) = P(A/\mathfrak{A})(v)$. Then, since F is $S\mathfrak{A}$-integrable, $\sim F : V \rightarrow R$ is a $\sim P$-integrable random variable on $(V, \mathcal{M}(\mathfrak{A}), \sim P)$, so, for an $S\mathfrak{A}$-integrable random variable $G : V \rightarrow * R$ on $(V, \mathfrak{A}, P)$ which satisfies (iii), we have that for $B \in \mathfrak{A}$
\[
\int \sim Gd\sim P = \text{st}(\sum (G(v)P(v), v \in B)) = \text{st}(\sum (F(v)P(v), v \in B)) = \int \sim Fd\sim P
\]
Hence, by the same arguments as in proof of (ii)
\[
G(v) \approx F(v) = P(A/\mathfrak{A})(v) \quad P\text{-n. s.}
\]
For $H : V \rightarrow * R$ which is $S\mathfrak{A}$-integrable and satisfies $\sum(|H(v) - F(v)|P(v), v \in V) \approx 0$, like in (ii), we have that $\sim H(v) = \sim F(v)$ $P$-a.s. implies that for any $B \in \mathfrak{A}$
\[
\text{st}(\sum (H(v)P(v), v \in B)) = \int \sim Hd\sim P = \int \sim Fd\sim P
= \text{st}(\sum (F(v)P(v), v \in B)) = \text{st}(P(A \cap B))
\]
i.e.
\[
\sum (H(v)P(v), v \in B) \approx P(A \cap B)
\]
In [10] it is proved that for S-integrable random variable $X : V \rightarrow * R$ on $(V, * \mathfrak{B}(V), P)$ the internal conditional expectation $E(X/\mathfrak{A})$ is a lifting of $\sim E(\sim X/\mathcal{M}(\mathfrak{A})), \sim E(\sim X/\mathcal{M}(\mathfrak{A}))$ being the conditional expectation of $\sim X : V \rightarrow R$ relative to sub-$\sigma$-algebra $\mathcal{M}(\mathfrak{A}) \subseteq \mathcal{M}(\sim P)$. From this result we derive the following nonstandard characterization of the conditional probability $\sim P(A/\mathcal{M}(\mathfrak{A})), A \in \mathcal{M}(\sim P)$ on a Loeb space.
THEOREM 5. Let \((V, \mathcal{P}(V), P)\) be a hyperfinite probability space, \(\mathfrak{A}\) an internal subalgebra of \(\mathcal{P}(V)\) and \(\sim P(A/\mathfrak{M}(\mathfrak{A}))\) the conditional probability of \(A \in \mathfrak{M}(\sim P)\) relative to sub-\(\sigma\)-algebra \(\mathfrak{M}(\mathfrak{A}) \subseteq \mathfrak{M}(\sim P)\). Then there exists a set \(B \in \mathcal{P}(V)\) such that
\[
\text{st}(P(B/\mathfrak{A})) = \sim P(A/\mathfrak{M}(\mathfrak{A})) \quad P\text{-a.s.}
\]

Proof. For \(A \in \mathfrak{M}(\sim P)\) there is a set \(B \in \mathcal{P}(V)\) such that \(\sim P(A\Delta B) = 0\).

The indicator function \(I_B\)
\[
I_B(v) = \begin{cases} 
1, & v \in B \\
0, & v \notin B
\end{cases}
\]
is an internal, \(S\)-integrable random variable on \((V, \mathcal{P}(v), P)\). Since
\[
P \{ v \mid I_A(v) \neq I_B(v) \} = P(A\Delta B) = 0
\]
\(I_B\) is an \(S\)-integrable lifting of \(I_A\). In view of [10], this implies
\[
\sim E(I_B/\mathfrak{A}) = \sim E(I_A/\mathfrak{M}(\mathfrak{A})) \quad P\text{-a.s.} \quad \text{and so}
\]
\[
\text{st}(P(B/\mathfrak{A})) = \sim P(A/\mathfrak{M}(\mathfrak{A})) \quad P\text{-a.s.}
\]

REFERENCES


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