A PROPERTY BETWEEN COMPACT AND STRONGLY COUNTABLY COMPACT

Dušan Milovančević

Abstract. In this paper we consider a class of spaces called hypercountably compact (hcc) spaces. The class of countably compact and the class of strongly countably compact (scc) spaces contain the class of hypercountably compact spaces. In Example 2.1, we give a strongly countably compact space which is not hypercountably compact. In the class of spaces satisfying the first axiom of countability the notions hcc and scc coincide (Theorem 2.3). Some equivalent conditions for a space to be hcc are given in Theorem 2.2. The hcc property is not a continuous invariant (Example 2.4). In Section 3 we consider compact spaces which contain noncompact hcc (scc) spaces as subspaces. In Section 4 we also consider strongly sequentially compact (scc) spaces.

1. Introduction

The closure of a subset A of a space X is denoted by cl_X(A). In this paper we assume that all spaces are Hausdorff (T_2-spaces). For notations and definitions not given here see [1], [3], [5].

Definition 1.1. [4]. A space X is strongly countably compact (scc) if every countable subset in X has a compact closure in X.

Definition 1.2. A space X is hypercountably compact (hcc) if the union of every countable family of compact sets in X has a compact closure in X.

A topological space X is called countably compact if every countable open cover of X has a finite subcover.

The following diagram illustrates the implications among the compactness properties that we consider:

\[
\text{compactness} \rightarrow \text{hypercountably compactness} \rightarrow \text{strongly countably compactness} \rightarrow \text{countably compactness}
\]

Counterexamples show that the implications in the diagram are not reversible. That countable compactness does not imply strongly countable compactness, see

AMS Subject Classification (1980): Primary 54D30, 54B20.
In example 2.1 the space is strongly countably compact but it is not hypercountably compact. Let \([0, \omega_1)\) be the space of ordinals less than the first uncountable ordinal with the order topology. The space \([0, \omega_1)\) is hypercountably compact but is not compact: Let \(K = \{K_n : n \in \mathbb{N}\}\) be a countable family of compact sets in \([0, \omega_1)\). Let \(a_n = \sup(K_n), n \in \mathbb{N}\). Since the least upper bound of any countable subset of \([0, \omega_1)\) is countable and will be strictly less than \(\omega_1\), there exists \(a \in [0, \omega_1)\), \(a = \sup(\{a_n \in [0, \omega_1) : n \in \mathbb{N}\})\) and is strictly less than \(\omega_1\). The set \(K = [0, a]\) is compact in \([0, \omega_1)\) and \(K_n \subset K\) for all \(n \in \mathbb{N}\). Hence, by definition 1.2, the space \([0, \omega_1)\) is hypercountably compact. The space \([0, \omega_1)\) fails to be compact since the collection \(\{(0, x) : x > \omega_1\}\) is an open cover with no finite subcover.

Let \(X\) be a topological \(T_2\)-space. Then:

1. \(\exp(X)\) denotes the space of all non-empty closed subsets of \(X\) with finite topology. The finite topology on \(\exp(X)\) is the one generated by open collection on the form \(\{U_1, U_2, \ldots, U_n\} = \{F \in \exp(X) : F \subseteq \bigcup_{i=1}^{n} U_i\} \) and \(F \cap U_i \neq \emptyset\) for \(i = 1, 2, \ldots, n\) where \(U_1, U_2, \ldots, U_n\) are open subsets of \(X\). The finite topology on \(\exp(X)\) is also known as the exponential topology or the Vietoris topology. The space \(\exp(X)\) is known as the hyperspace of \(X\) (see [9], [10], [11]).

2. \(\mathcal{K}(X)\) denote the set of all non-empty compact subsets of \(X\) as a subspace of \(\exp(X)\).

3. \(\mathcal{K}^{(2)}(X) = \mathcal{K}(\mathcal{K}(X)), \ \mathcal{K}^{(n)}(X) = \mathcal{K}(\mathcal{K}^{(n-1)}(X)), n = 2, 3, \ldots\)

4. \(\mathcal{F}_n(X) = \{F \subseteq X : F\) has at most \(n\) points\} \(\subseteq \mathcal{K}(X)\).

5. \(\mathcal{F}(X) = \{F \subseteq X : F\) is finite\} \(\subseteq \mathcal{K}(X)\).

Definition 1.3. [1] A quasi-ordered and directed set \((S, \preceq)\) is \(\mathbb{N}\)-directed if for every countable subset \(S_0 = \{s_1, s_2, \ldots, s_n, \ldots\} \subseteq S\) there exists an \(s \in S\) such that \(s_n \preceq s\), for all \(n \in \mathbb{N}\).

2. Relationship between hcc and scc spaces and some properties of hcc spaces

It is clear that every hcc space is a scc space. The following example shows that not every strongly countably compact space is hypercountably compact.

Example 2.1. (A strongly countably compact space which is not hypercountably compact). Let \([0, \omega_1)\) \(([0, \omega_1])\) be the the space of ordinals less than or equal to the first uncountable ordinal (first countable ordinal) with the order topology. Let \(X_1 = [0, \omega_1) \times [0, \omega_0)\). Since ordinal space \([0, a] (a is any ordinal) is compact and Hausdorff, so is the Tychonoff product \(X_1\). Since every compact Hausdorff space is normal, \(X_1\) is normal. Let \(X_2 = [0, \omega_1) \times [0, \omega_0) - \{(\omega_1, n) : n \in [0, \omega_0)\}\) be subspace of \(X_1\). Then \(X_2\) is noncompact and normal in the subspace topology. Furthermore, \(X_2\) is hypercountably compact: Let \(D = \{K_n \in \mathcal{K}(X_2) : n \in \mathbb{N}\}\) be a countable family of compact sets in \(X_2\) and let \((\omega_1, \omega_0) \not\in K_n\) for all \(n \in \mathbb{N}\). The sequences \(D' = \{K'_n \in \mathcal{K}([0, \omega_1)) : n \in \mathbb{N}\}\) and \(D'' = \{K''_n \in \mathcal{K}([0, \omega_1)) : n \in \mathbb{N}\}\) are projections of \(D\) onto \([0, \omega_1]\) and \([0, \omega_0]\). Sets \(K'_n\) and \(K''_n\) are compact and
$K'_n \subseteq [0, \omega_1)$, $K''_n \subseteq [0, \omega_1)$ for all $n \in N$. Let $a_n = \sup(K'_n)$, $n \in N$. Since the least upper bound of any countable subset of $[0, \omega_1)$ is countable and is strictly less than $\omega_1$ there exists $a \in [0, \omega_1)$, $a = \sup(\{a_n \in [0, \omega_1) : n \in N\})$ and strictly less than $\omega_1$. The set $K = [0,a] \times [0,\omega_1]$ is compact in $X_2$ and $K_n \subseteq K$ for all $n \in N$. If $(\omega_1, \omega_2) \in K_n$ for infinite number of members $K_n \in \mathcal{D}$, then the set $K' = [0,a] \times [0,\omega_1] \cup ([0,\omega_1] \times \{\omega_2\})$ is compact in $X_2$ and $K_n \subseteq K'$ for all $n \in N$. Hence $X_2$ is a hypercountably compact space. Let $X_3 = X_2 \cup \{p\}, (p \notin X_2)$ be the one-point compactification of $X_2$. So we get the following diagram

$$(0, \omega_0) \ldots \ldots (\omega_1, \omega_0)$$

$$(0,0) \ldots \ldots p \ldots \ldots (\omega_1, 0)$$

which we will call the space $X_3$. Then:

1. $X_3$ is compact and a $T_1$ space.
2. $X_3$ is not Hausdorff since the point $p$ and $(\omega_1, \omega_0)$ have no disjoint neighborhoods.
3. The point $(\omega_1, \omega_0)$ is an accumulation (limit) point of $X_3$.
4. Let $X_4 = X_3 = \{(\omega_1, \omega_0)\}$ be a subspace of $X_3$. The set $A = \{(a, \omega_1) : 0 \leq a < \omega_1\}$ is a subset of $X_4$. Furthermore, the set $A$ is a closed subset of $X_4$ homeomorphic to $[0, \omega_1)$. Hence $X_4$ is not compact since the space $[0, \omega_1)$ is not compact.
5. The space $X_4$ is Hausdorff but it is not regular since the point $p$ and the set $A = \{(a, \omega_1) : 0 \leq a < \omega_1\}$ have no disjoint neighborhoods.
6. Let $A = \{a_n \in X : n \in N\}$ be any countable subset of $X_4$ and let $p \in A$. Then $A = \{(x_n, y_n) : x_n \in [0, \omega_1), y_n \in [0, \omega_1); n \in N\}$ where $\{x_n \in [0, \omega_1) : n \in N\} \subseteq [0, \omega_1)$ and $\{y_n \in [0, \omega_1) : n \in N\} \subseteq [0, \omega_1)$. Let $a$ be an upper bound for the $x_n$; $a < \omega_1$ since $\omega_1$ has uncountably many predecessors, while $a$ has only countably many. Thus the set $[0,a) \times [0, \omega_1)$ is closed and compact in $X_4$. Furthermore, $A \subseteq [0,a) \times [0, \omega_1)$ and $\text{cl}_{X_4}(A) \subseteq [0,a) \times [0, \omega_1)$ and $\text{cl}(X_4)(A)$ is a compact subset of $X_4$. Hence $X_4$ is a strongly countably compact space.
7. The space $X_4$ is not hypercountably compact because there exists a countable family $\mathcal{F} = \{([0, \omega_1) \times \{k\}) \cup \{p\} : k = 0,1,2,\ldots\}$ of non-empty compact subsets of family $X_4$ such that:

$$\text{cl}_{X_4}(\mathcal{F}) = \text{cl}_{X_4}(\bigcup\{([0, \omega_1) \times \{k\}) \cup \{p\} : k = 0,1,\ldots\}) = X_4.$$  Hence the space $X_4$ is scc but not hcc.

The following result gives a characterization of hcc spaces.

**Theorem 2.2.** Let $X$ be a $T_2$-space. The following are equivalent:

1. $X$ is hypercountably compact,
2. $\mathcal{K}(X)$ is $\aleph_0$-directed by inclusion,
(3) $\mathcal{K}(X)$ is countably compact,

(4) $\mathcal{K}(X)$ is strongly countably compact,

(5) $\mathcal{K}(X)$ is hypercountably compact,

(6) $\mathcal{K}^{[m]}(X)$ is hypercountably compact.

Proof. It is clear that (1) $\Leftrightarrow$ (2). Let $\{K_n \in \mathcal{K}(\mathcal{K}(X)) : n \in N\}$ be a countable family of compact sets in $\mathcal{K}(X)$. Then $|\mathcal{K}_n| = \bigcup\{K \in \mathcal{K}_n \} \in \mathcal{K}(X')$ for all $n \in N$. Since $X$ is a hcc space then $\text{cl}_X(\bigcup\{\mathcal{K}_n : n \in N\}) \in \mathcal{K}(X)$ which implies that $\exp(\text{cl}_X(\bigcup\{\mathcal{K}_n : n \in N\}))$ is a compact subset of $\mathcal{K}(X)$ and $\mathcal{K}_n \subseteq \exp(\text{cl}_X(\bigcup\{\mathcal{K}_n : n \in N\}))$ for all $n \in N$. Hence $\mathcal{K}(X)$ is a hcc space and (1) $\Leftrightarrow$ (5).

(4) $\Rightarrow$ (1): Suppose that $\mathcal{K}(X)$ is a scc space and let $\mathcal{K}\{K_n \in \mathcal{K}(X) : n \in N\}$ be any countable family of compact subsets of $X$. Then $\text{cl}_{\mathcal{C}(\mathcal{K}(X))}(\mathcal{K})$ is a compact subspace of $\mathcal{K}(X)$. Since $\text{cl}_{\mathcal{C}(\mathcal{K}(X))}(\mathcal{K})$ is compact we have (see [11]) $|\text{cl}_{\mathcal{C}(\mathcal{K}(X))}(\mathcal{K})| = \bigcup\{K \in \text{cl}_{\mathcal{C}(\mathcal{K}(X))}(\mathcal{K})\}$ is a compact subset of $X$ and $\mathcal{K}_n \subseteq |\text{cl}_{\mathcal{C}(\mathcal{K}(X))}(\mathcal{K})| = \bigcup\{K \in \text{cl}_{\mathcal{C}(\mathcal{K}(X))}(\mathcal{K})\}$ for all $n \in N$. Hence $X$ is a hcc space.

(1) $\Rightarrow$ (4): Since (1) $\Rightarrow$ (5) and (5) $\Rightarrow$ (4) we have (1) $\Rightarrow$ (4).

(5) $\Rightarrow$ (1): Since (5) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) we have (5) $\Rightarrow$ (1).

(5) $\Leftrightarrow$ (6): Can be obtained in similar way.

(3) $\Rightarrow$ (5): Let $\mathcal{K}(X)$ be countably compact and let $\mathcal{K} = \{K_n \in \mathcal{K}(X) : n \in N\}$ be a countable family of $\mathcal{K}(X)$ increasing by inclusion. Since $\mathcal{K}(X)$ is countably compact there exists a $K \in \mathcal{K}(X)$ such that $K$ is an accumulation point of $\mathcal{K}$. Let $F = \text{cl}_X(\bigcup\{K_n \in \mathcal{K}\}) = \text{cl}_X(\bigcup\{K_n \in \mathcal{K}(X) : n \in N\})$ and let $\langle U_1, U_2, \ldots, U_m \rangle$ be a basic open set containing $F$. Then $F \subseteq \bigcup_{i=1}^m \bigcup_{j=1}^m \langle U_i \rangle$ and $F \cap U_i \neq \emptyset$ for $i \in \{1, 2, \ldots, m\}$. Then for each $n \in N$ we have that $K_n \subseteq \bigcup_{i=1}^m \langle U_i \rangle$ and for all $i \in \{1, \ldots, m\}$ there exists an $n_i \in N$ such that $U_i \cap K_{n_i} \neq \emptyset$. Let $n_0 = \max\{\{n_i : i \in \{1, 2, \ldots, m\}\}\}$. Then for each $n > n_0$ we have $K_n \cap U_i \neq \emptyset$, $i \in \{1, 2, \ldots, m\}$ and it follows that $K_n \in \langle U_1, U_2, \ldots, U_m \rangle$. Hence $F$ is an accumulation point of $\mathcal{K}$. Since $\mathcal{K}(X)$ is countably compact, $F \in \mathcal{K}(X)$. Furthermore; for each $K \in \mathcal{K}(X)$, $K \neq F$, we have that $K$ is not an accumulation point of $\mathcal{K}$. Let $K \neq F$. Then there is a point $x \in K$ and $x \notin F$ of there is a is a point $x \in F$ and $x \notin K$.

Case I: $x \in K$ and $x \notin F$. Since $F \subseteq X$ is compact and $X$ in Hausdorff, there exists an open set $U_x$ containing $x$ such that $U_x \cap U = \emptyset$. Then $F \notin \langle U_x \rangle$ and $K \notin \langle U_x \rangle$. Furthermore, for each $n \in N$, we have $K_n \notin \langle U_x \rangle$ and $K \notin \langle U_x \rangle$. Hence $K$ is not an accumulation point of $\mathcal{K}$.

Case II: $x \in F$ and $x \notin K$. Since $X$ is Hausdorff and $K, F$ are compact subsets of $X$, there exist open neighborhoods $U$ of $K$ and $U$ of $x$ such that $U \cap U_x = \emptyset$. Then $K \notin \langle U \rangle$ and $F \notin \langle U \rangle$ which implies that $K_n \notin \langle U \rangle$ for all $n \in N$. Hence $K$ is not an accumulation point of $\mathcal{K} = \{K_n \in \mathcal{K}(X) : n \in N\}$. Since $F = \text{cl}_X(\bigcup\{K_n \in \mathcal{K}(X) : n \in N\}$.

* $\langle U \rangle$ stands for $\{F \in \exp(X) : F \cap U \neq \emptyset\}$. 
$K) = \text{cl}_X(\bigcup\{K_n \in \mathcal{K}(X) : n \in N\})$ is contained in $\mathcal{K}(X)$ by 2.1, the space $X$ is hypercountably compact.

Suppose that $\mathcal{K} = \{K_n \in \mathcal{K}(X) : n \in N\}$ is not increasing by inclusion. Then the family $\mathcal{K}' = \{K'_n \in \mathcal{K}(X) : n \in N\}$ with $K'_n = \bigcup_{i=1}^{n} K_i$ is increasing by inclusion and, by the foregoing, there exists a compact set $F''$ of $X$ such that $F'' = \text{cl}_X(\bigcup\{K' \in \mathcal{K}'\})$. Since for each $n \in N$ we have $K_n \subseteq K'_n$, $F = \text{cl}_X(\bigcup\{K_n \in \mathcal{K}\})$ is a compact subset of $X$. Hence $X$ is hypercountably compact. This completes the proof.

**Theorem 2.3.** Let $X$ be a first countable strongly countably compact $T_2$-space. The $X$ is a hypercountably compact space.

**Proof.** To prove 2.3, it suffices by 2.2, to show that $\mathcal{K}(X)$ is a scc space. Since $X$ is a first countable $T_2$-space, then $\mathcal{K}(X)$ is also a first countable $T_2$-space (see [11]). Let $\mathcal{F}$ denote the family of all finite subsets of $X$. Then $\mathcal{F} \subset \mathcal{K}(X)$ and $\text{cl}_{\mathcal{K}(X)}(\mathcal{F}) = \mathcal{K}(X)$ (see [9], [11]). Let $\mathcal{K}$ be any countable family of compact sets in $X$, $\mathcal{K} = \{K_n \in \mathcal{K}(X) : n \in N\}$. Since $\mathcal{K}(X)$ is a first countable space and $\text{cl}_{\mathcal{K}(X)}(\mathcal{F}) = \mathcal{K}(X)$, then for each $K_n \in \mathcal{K}$ there exists $A[n] = \{A^n_k : k \in N\}$ such that $K = \text{cl}_{\mathcal{K}(X)}(A[n])$ ($K = \{\{ x \} : x \in K_n\}$ is a compact subset of $\mathcal{K}(X)$ homeomorphic to $K_n$ for all $n \in N$. The family $\mathcal{A} = \{A^n_k : (k, n) \in N \times N\}$ is also a countable family and $|\mathcal{A}| = \bigcup_1^N (A^n_k : (k, n) \in N \times N)$ is also a countable subset in $X$. Since $X$ is a strongly countably compact space, then $K = \text{cl}_X(\bigcup_1^N)\mathcal{A}$ is a compact subset of $X$ and $\exp(K)$ is also a compact subset of $\mathcal{K}(X)$). Now we have $K \in \exp(K) \subseteq \mathcal{K}(X)$, and $\exp(K)$ is a compact subset of $\mathcal{K}(X)$ which implies that $\text{cl}_{\mathcal{K}(X)}(\mathcal{K}) \subseteq \exp(K)$. Hence $\text{cl}_{\mathcal{K}(X)}(\mathcal{K})$ is a compact subset of $\mathcal{K}(X)$ and $\mathcal{K}(X)$ is an scc space. This completes the proof.

It can be shown that every continuous image of an scc space is an scc space. The following example shows that the continuous image of a hcc need not be an hcc space.

**Example 2.4.** Let $X = [0, \omega_1] \times [0, \omega_1] - \{ (\omega_1, x) \} \subseteq [1, \omega_1]$ be the subspace of the product $[0, \omega_1] \times [0, \omega_1]$. Then, by example 2.1, $X$ is normal and hypercountably compact in the subspace topology. Let $Y = X_4$ where $X_4$ is the space in example 2.1. The mapping $f : X \to Y$ is defined by $f(x) = x$, for all $x \neq (\omega_1, 0)$ and $f(\omega_1, 0) = p$ is continuous surjection. The space $X$ is hcc, but $Y$ is not a hcc space.

**Remark.** A continuous surjection $f : X \to Y$ is a compact-covering if whenever $B$ is a compact set in $Y$, there exits a compact set $A$ in $X$ such that $(A) = B$. If $f : X \to Y$ is a compact-covering mapping and $X$ is a hcc space, then $Y$ is hcc.

We now state, without proof, some properties of hcc spaces.

**Proposition 2.5.** A closed subset $A$ of a hcc space $X$ is itself hcc.

**Proposition 2.6.** Each hcc subset of a first countable $T_2$-space $X$ is closed in $X$. 


Example 2.7. Let $X = [0, \omega_1]$ and $A = [0, \omega_1) \subset X$. The subset $A$ is hcc but $A$ is not closed in $X$. The space $X = [0, \omega_1]$ is a $T_2$-space but is not first countable.

**Proposition 2.8.** Let $(X_a : a \in A)$ be a family of non-empty spaces. Then the product space $X = \prod \{x_a : a \in A\}$ is hcc if and only if $X_a$ is hcc for each $a \in A$.

3. P-points, weak P-points and some characterizations

**Definition 3.1.** Let $X$ be a topological $T_2$-space.

(1) A point $p \in X$ is said to be a P-point provided that each intersection of countably many neighbourhoods of $p$ is a neighbourhood of $p$.

(2) A point $p \in X$ is a weak P-point if $p \not\in \text{cl}_X(F)$ for each countable $F \subset X - \{p\}$.

It is easy to see that every P-point is a weak P-point. In example 2.1, the point $(\omega_1, \omega_0)$ is a weak P-point but is not a P-point.

**Theorem 3.2.** Let $X$ be a compact $T_2$-space and $p \in X$ be a limit point in $X$. Then:

(1) The subspace $X - \{p\}$ is an scc space if and only if the point $p$ is a weak P-point.

(2) The subspace $X - \{p\}$ is a hcc space if and only if the point $p$ is a P-point in $t$.

**Proof.** $X - \{p\}$ is an open noncompact subset of $X$.

(1) Suppose that $X - \{p\}$ is an scc space and let $A \subset X - \{p\}$ be a countable set. Then $\text{cl}_{X - \{p\}}(A) = \text{cl}_X(A)$ is a compact set in $X - \{p\}$ and $p \not\in \text{cl}_X(A)$. Hence point $p$ is a weak P-point in $X$.

Conversely, suppose that $p$ is a weak P-point in $X$. Let $A \subset X \{p\}$ be a countable set. Then $p \not\in \text{cl}_X(A)$ which implies that $\text{cl}_X(A) = \text{cl}_{X - \{p\}}(A)$. Since $\text{cl}_X(A)$ is a compact (the space $X$ is compact and Hausdorff) subset in $X$, then $\text{cl}_{X - \{p\}}(A)$ is a compact subset in $X - \{p\}$. Hence, subspace $X - \{p\}$ is an space.

(2) Suppose that $X - \{p\}$ is an hcc space and let $\{U_n(p) : n \in N\}$ be any countable family of open neighbourhoods of the point $p$ in $X$. Then $\bigcap \{U_n(p) : n \in N\} = \bigcup \{X - U_n(p) : n \in N\}$ and $X - U_n(p) \subset X - \{p\} \subset X$ is a compact set for all $n \in N$. Since $X - \{p\}$ is an hcc space there exists a compact set $K \subset X - \{p\}$ such that $X - U_n(p) \subset K$ for all $n \in N$. The set $U = X - K$ is an open neighbourhood of $p$ in $X$. Therefore, $X - K \subset \bigcap \{U_n(p) : n \in N\}$. Hence $p$ is a P-point in $X$.

Conversely, suppose that $p$ is a P-point in $X$ and let $\{K_n : n \in N\}$ be any countable family of compact sets in $X - \{p\}$. Then $\bigcap \{K_n : n \in N\} = \bigcap \{X - K_n : n \in N\}$ and $X - K_n$ is an open neighbourhood of $p$ in $X$ for all $n \in N$. Since $p$ is a P-point in $X$, there exists an open neighbourhood $U$ of $p$ such that $U \subset \bigcap \{X - K_n : n \in N\}$. Therefore, $K_n \subset X - U$ for each $n \in N$ and $X - U$ is a compact subset of $X - \{p\}$. Hence, by definition 1.2, subspace $X - \{p\}$ is an hcc space. This completes the proof.
COROLLARY 3.3. Let $X$ be a compact $T_2$-space and $p \in X$ be a limit point in $X$. If $P$ is a weak $P$-point which is not a $P$-point, then the subspace $X - \{p\}$ is a strongly countably compact space which is not a hypercountably compact space.

Example 3.4. Let $\beta N$ denote the Stone-Čech compactification of positive integers $N$ and $N^* = \beta N - N$. Kunen in [7] has shown using Martin’s Axiom that there exists a weak $P$-point $p$ in $N^*$ which is not a $P$-point. By corollary 3.3, the space $X = N^* - \{p\}$ is an example of strongly countably compact space which is not hypercountably compact space.

4. Strongly sequentially compact (ssr) spaces

Definition 4.1. A space $X$ is strongly sequentially compact (ssc) if $\mathcal{K}(X)$ is sequentially compact.

Proposition 4.2. Every strongly sequentially compact space is sequentially compact.

Proof. The subspace $X = \{x \in \mathcal{K}(X) : x \in X\} \subset \mathcal{K}(X)$ is a closed subset of the sequentially compact $\mathcal{K}(X)$ and $X$ is homeomorphic to $X$. Since $\mathcal{K}(X)$ is sequentially compact, so is $X$.

The converse is not necessarily true. There exist spaces which are sequentially compact but not strongly sequentially compact. Such an example will be given after the following lemma.

Lemma 4.3. [1]. If a compact space $Y$ is a continuous image of the remainder $cX - z(X)$ of a compactification $cX$ a locally compact space $X$, then the space $X$ has a compactification $cX \leq cX$ with the remainder homeomorphic to $Y$.

Applying transfinite induction we get a continuous mapping $f : \beta N - N \rightarrow [0, \omega_1]$ onto the space $[0, \omega_1]$ of all ordinal numbers less than or equal to the first uncountable ordinal $\omega_1$ [1, p. 296].

Applying Lemma 3.3, there exists a compactification $gN$ of the space $N$ whose remainder $gN - N$ coincides with $[0, \omega_1]$. The space $gN - \omega_1$ is a separable, locally compact, first countable and sequentially compact normal space which is not a compact space. Since the space $X = gN - \{\omega\}$ is separable it is not strongly countably compact. Suppose that $\mathcal{K}(X)$ is sequentially compact. Then $\mathcal{K}(X)$ is countably compact and by Theorem 2.2, $X = gN - \{\omega_1\}$ would be hypercountably compact which is impossible. Therefore, $X$ is an example of a sequentially compact space, which is not strongly sequentially compact.

Remarks. (a) The space $\beta N$ is not sequentially compact. This can be also proved directly: to sequence $1, 2, \ldots$, of point of $\beta N$ does not contain any convergent subsequence, since for every increasing sequence $k_1 < k_2 < \ldots$ of positive integers the sets $A = \{k_1, k_3, \ldots\}$ and $B = \{k_2, k_4, \ldots\}$ have disjoint closures in $\beta N$ and this implies that the sequence $k_1, k_2, k_3, \ldots$ does not converge.
(b) By Proposition 2.8 the space \( X = \beta N \times [0, \omega_1) \) is a hypercountably compact but is not sequentially compact because \( \beta N \times \{ x_0 \}, \ x_0 \in [0, \omega_1) \) is a closed subspace of \( X \) homeomorphic to \( \beta N \).

(c) The space \( X = \beta N \times X_4 \) (\( X_4 \) is the space in example 2.1) is strongly countably compact which is not hypercountably compact. By (b) the space \( X \) is not sequentially compact.

**Theorem 4.4.** Every strongly sequentially compact space is hypercountably compact.

*Proof.* Let \( \mathcal{K}(X) \) be a sequentially compact space. Then it is countably compact. By Theorem 2.2, \( X \) is an hcc space. This completes the proof.

**Proposition 4.5.** A closed subset of a strongly sequentially compact space is itself strongly sequentially compact.

*Proof.* Let \( X \) be strongly sequentially compact and \( Y \) be a closed subset of \( X \). Clearly \( \mathcal{K}(Y) = \mathcal{K}(X) \cap \exp(Y) \). By [11], \( \exp(Y) \) is a closed subset of \( \exp(X) \). Hence \( \mathcal{K}(Y) \) is a closed subset of \( \mathcal{K}(X) \) and \( \mathcal{K}(Y) \) is sequentially compact. By definition 4.1, \( Y \) is strongly sequentially compact. This completes the proof.

*Remarks.* (a) The space \( \beta N \) is compact (hypercountably compact) but not strongly sequentially compact.

(b) The space \( X = [0, \omega_1) \) is strongly sequentially compact but not compact.

(c) The space \( X = [0, \omega_0] \) is both compact and strongly sequentially compact.

The following diagram illustrates the implications which exist among the properties of compactness that we consider:
REFERENCES


Mašinski fakultet Niš
1800 Niš, Beogradska 14
Jugoslavija

(Received 09 01 1985)