ON THE MINIMAL DISTANCE OF THE ZEROS OF A POLYNOMIAL

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1. Let

\[ p(x) = \sum_{\nu=0}^{n} a_{\nu} x^\nu, \quad (a_{\nu} \in C, \ a_n \neq 0) \]

be a complex polynomial whose zeros \( x_1, \ldots, x_n \) are mutually distinct. In this paper we give a method of finding some positive lower bounds of

\[ \min_{i \neq j} |x_i - x_j|. \]

2. In the sequel we shall use some well known facts about polynomials. Let \( p(x) = a_0 + \cdots + a_n x^n \) \((a_n \neq 0)\) be any complex polynomial. There are many known formulas ([1], [2]) of the type

\[ |x_i| \leq M \quad (i = 1, \ldots, n) \]

where \( x_1, \ldots, x_n \) are all zeros of \( p(x) \) and \( M \) is a positive constant. So, a classical result due to Cauchy [1] is

\[ |x_i| \leq 1 + \max_{1 \leq \nu < n} (|a_\nu|/|a_n|) \]

We emphasize that in this case, and the same is almost ever, \( M \) has the following porperty

\[ M \text{ is an increasing function in each } |a_0|/|a_n|, \ldots, |a_{n-1}|/|a_n| \]

Let, further, besides \( p(x) \)

\[ p_1(x) = b_0 + \cdots + b_m x^m, \quad (b_m \neq 0) \]

AMS Subject Classification (1980): Primary 12D10, 26C10, 30C15.
be another complex polynomial. Then there is a polynomial of the form
\begin{equation}
    r(x) = c_0 + \cdots + c_{n-1}x^{n-1}
\end{equation}
such that the equality
\begin{equation}
    p_i(x_i) = r(x_i)
\end{equation}
holds for every zero \( x_i \) of the polynomial \( p(x) \). In other words we have the following relation
\begin{equation}
    p_i(x) \equiv r(x) \pmod{p(x)}
\end{equation}
There are at least two methods of finding \( r(x) \): by the division algorithm or by applying, enough number of times, the substitution
\[ x^n \rightarrow -a_n^{-1}(a_0 + a_1x + \cdots + a_{n-1}x^{n-1}). \]
Note that for \( r(x) \) we shall also use the notation \( r(p_1(x), p(x)) \).

Suppose now that we would like to have a polynomial \( r(x) \) of the form (6) such that the equality
\begin{equation}
    1/p_i(x_i) = r(x_i)
\end{equation}
holds for every zero \( x_i \) of \( p(x) \). Generally such a polynomial \( r(x) \) does not exist.
It exists just in the case the polynomials \( p(x) \) and \( 1 \) have no common zero, i.e. they are relatively prime polynomials. Then \( r(x) \) can be found by the Euclidean algorithm, for example. Namely, in such a way we can find two polynomials \( e_1(x), e_2(x) \) such that the identity
\[ e_1(x)p(x) + e_2(x)p_1(x) = 1 \]
holds. Hence we have the equality \( 1/p_1(x_i) = e_2(x_i) \) and consequently for \( r(x) \) we may take the polynomial \( r(e_2(x), p(x)) \). Note that for the obtained \( r(x) \), i.e. \( r(e_2(x), p(x)) \) we shall also use the notation as before; \( r(1/p_1(x), p(x)) \). More generally, if \( a(x)/b(x) \) is any rational function, where \( b(x) \) and \( p(x) \) are relatively prime polynomials, then by \( r(a(x)/b(x), p(x)) \) will be denoted a polynomial of the form (6) such that the equality \( a(x_i)/b(x_i) = r(x_i) \) holds for every zero \( x_i \) of the polynomial \( p(x) \). Obviously the polynomial \( r(x) \) a unique.

Example 1. Let
\begin{equation}
    p(x) = x^3/3 - x^2 + 2x + 1/3, \quad p_1(x) = x^2 - 2x + 2.
\end{equation}
Then using the Euclidean algorithm we obtain the following polynomial equalities
\[ p(x) = p_1(x)(x - 1)/3 + (2x + 3)/3, \quad p_1(x) = (2x + 3)/3 \cdot (3x/2 - 21/4) + 29/4 \]
from which on eliminating the polynomial \( (2x + 3)/3 \) we infer the equality
\[ p(x)(-6x + 42)/29 + p_1(x)(2x^2 - 9x + 11)/29 = 1 \]
Thus we see that
\[ r(1/p_1(x), p(x)) = (2x^2 - 9x + 11)/29 \]

3. Now we are going to describe, step by step, a method of finding a lower bound of (2) for a given polynomial (1).

Firstly, we begin with the Taylor formula
\[ p(x) = p(x_i) + (x - x_i)p'(x_i) + \cdots + (x - x_i)^n p^{(n)}(x_i) \]
where \( i \in \{1, \ldots, n\} \) is fixed. Hence we conclude that the following equation in \( d \)
\[ d^{n-1} \cdot p'(x_i) + d^{n-2} \cdot p''(x_i) + \cdots + d \cdot \frac{p^{(n-1)}(x_i)}{(n - 1)!} + \frac{p^{(n)}(x_i)}{n!} = 0 \]
has the zeros \( (x_1 - x_i)^{-1}, \ldots, (x_{i-1} - x_i)^{-1}, (x_{i+1} - x_i)^{-1}, \ldots, (x_n - x_i)^{-1} \).

Secondly, let
\[ M(|p^{(n)}(x_i)/n!p'(x_i)|, |p^{(n-1)}(x_i)/(n - 1)!p'(x_i)|, \ldots, |p''(x_i)/2!p'(x_i)|) \]
be any increasing (in the sense of (5)) upper bound of the moduli of the zeros of the equation (11). Thus, we have the inequality
\[ |x_j - x_i| \leq M(|p^{(n)}(x_i)/n!p'(x_i)|, |p^{(n-1)}(x_i)/(n - 1)!p'(x_i)|, \ldots, |p''(x_i)/2!p'(x_i)|) \]
\[ \ldots, |p''(x_i)/2!p'(x_i)|) \]

Thirdly, suppose that a constant \( A > 0 \) is an upper bound of \( |x_i| \) (\( i = 1, \ldots, n \)).

Fourthly, suppose that we have determined the following polynomials
\[ r(p^{(n)}(x)/n!p'(x)), r(p^{(n-1)}(x)/(n - 1)!p'(x)), \ldots, r(p''(x)/2!p'(x)) \]
which exist since \( p(x) \) has mutually distinct zeros. Denote these polynomials by \( r_n(x), r_{n-1}(x), \ldots, r_2(x) \) respectively.

For any polynomial \( f(x) = f_0 + f_1x + \ldots + f_x \) let \( |f|(x) \) denote the polynomial \( f_0| + |f_1|x + \cdots + |f_x|x^x \).

Fifthly, using the monotony of \( M \) and the inequalities \( |x_i| \leq A \) from (13) it follows that
\[ |x_j - x_i| \geq (M(|r_n|(A), |r_{n-1}|(A), \ldots, |r_2|(A)))^{-1} \quad (i \neq j) \]
which yields our final result.
THEOREM. The minimal distance of the zeros of the polynomial (1) satisfies the inequality

\[
\min_{j \neq i} |x_j - x_i| \geq (M(|r_n|(A), |r_{n-1}|(A), \ldots, |r_2|(A)))^{-1}
\]

Example 2. Let \( p(x) \) be the polynomial considered in Example 1. Then we have the following equalities

\[
p'(x) = p_1(x), \quad p''(x) = 2x - 2, \quad p'''(x) = 2
\]

As we have already established we have the equality (see (10))

\[
r(1/p'(x), p(x)) = (2x^2 - 9x + 11)/29
\]

In the next step we should decide which the \( M \)-formula to use. Let us take the Cauchy’s one. So, according to (13) we have the following inequality

\[
|x_j - x_i|^{-1} \leq 1 + \max(|p''(x_i)/2p(x_i)|, |p'''(x_i)/6p'(x_i)|)
\]

i.e. the inequality

\[
|x_j - x_i|^{-1} \leq 1 + \max(|x_i - 1/x_i^2 - 2x_i + 2|, |1/(x_i^2 - 2x_i + 2)|)
\]

Using (15) it is easily seen that

\[
\frac{1}{x^2 - 2x + 2} \equiv \frac{2x^2 - 9x + 11}{29} \pmod{p(x)},
\]

\[
\frac{x - 1}{x^2 - 2x + 2} \equiv \frac{-5x^2 + 8x - 13}{29} \pmod{p(x)}
\]

Thus the inequality of the type (14) reads

\[
\min_{i \neq j} |x_j - x_i| \leq 1/ \left( 1 + \max \left( \frac{2A^2 + 9A + 11}{29}, \frac{5A^2 + 9A + 13}{29} \right) \right)
\]

where \( A \) is an upper bound of \(|x_1|, |x_2|, |x_3|\). For instance, using the Cauchy formula (4) we conclude that

\[
\min_{i \neq j} |x_j - x_i| \geq 29/350,
\]

REFERENCES
