AN ORDERING OF THE SET OF SENTENCES OF PEANO ARITHMETIC

Aleksandar Ignjatović

Abstract. We consider a partial ordering of the set of sentences of Peano arithmetic $P$ induced by a theory $T$ extending $P$, which orders sentences according to the complexity of their "proofs". Using some properties of the ordering induced by the theory $P + \neg \text{Con}_P$ we prove that $P$ doesn’t have the Joint Embedding Property. We also describe models for $P$ which do not enrich the ordering induced by $P$, i.e. models satisfying $<_{\Sigma^1_0} \equiv <_P$, and we prove that for every consistent theory $T$, $T \supset P$, there is a theory $T' \supset P$ such that the ordering induced by the theory $T'$ is a linear extension of the ordering induced by the theory $T$.

By $L_P$ we denote the language of $P$ and by $S(P)$ the set of sentences of $P$. Any consistent extension of $P$ we denote by $T$, and $N$ stands for the structure of natural numbers. By $\mathcal{M}, \mathcal{N}, \ldots$ we denote nonstandard models for $P$, and by $|\mathcal{M}|, |\mathcal{N}|, \ldots$ their domains respectively. If $\mathcal{M}$ and $\mathcal{N}$ are models for $P$, then $\mathcal{M} \subset_{\Sigma^1_1} \mathcal{N}$ means that for all $\Sigma^1_1$-formulas $\varphi$ and all $a_0, \ldots, a_n \in |\mathcal{M}|$, $\mathcal{M} \models \varphi[a_0, \ldots, a_n]$ implies $\mathcal{N} \models \varphi[a_0, \ldots, a_n]$; similarly, we write $\mathcal{M} \subset_{\bar{\Sigma}^1_1} \mathcal{N}$ when $\mathcal{M} \models \varphi[a_0, \ldots, a_n]$ holds iff $\mathcal{N} \models \varphi[a_0, \ldots, a_n]$ holds.

We use the following model-theoretical consequence of Matijasevic’s theorem.

Lemma 0. Let $\mathcal{M}, \mathcal{N}$ be models for $P$; then $\mathcal{M} \subset \mathcal{N}$ implies $\mathcal{M} \subset_{\Sigma^1_1} \mathcal{N}$ and $\mathcal{M} \subset_{\bar{\Sigma}^1_1} \mathcal{N}$. □

For any sentences $\varphi$ and $\psi$ from $S(P)$, by $\varphi < \psi$ we denote the $\Sigma^1_1$-sentence

$$
\exists x (\text{Prf}_p(x, \bar{\varphi}) \land (\forall y < x) \neg \text{Prf}_p(y, \bar{\psi})).
$$

The following lemma enables us to introduce the ordering in $S(P)$.

Lemma 1. Let $T$ be a consistent extension of $P$; then the relation $<$ defined by $\varphi < \psi$ iff $T \vdash \varphi < \psi$ is transitive and irreflexive. □

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Since the sentence \( \varphi < \psi \) is \( \sum_1 \), it is obvious that \( \varphi <_P \psi \) holds iff \( \varphi < \text{Th}(N) \psi \) holds, i.e., \( \varphi < \text{Th}(N) \). The order type of \( \varphi <_P \) is \( \omega + \Lambda \); the set of theorems of \( P \) has order type \( \omega \), and \( \Lambda \) is the type of the empty ordering of the countable set of sentences from \( S(P) \) not provable in \( P \).

The ordering induced by the theory of a model \( \mathcal{M} \models P \) is linear iff \( \mathcal{M} \models \neg \text{Con}_P \). If \( \mathcal{M} \models \text{Con}_P \), then the ordering consists of a linearly ordered set of sentences having “proofs” in \( \mathcal{M} \), i.e. sentences satisfying \( \mathcal{M} \models \text{Thm}_P(\varphi \gamma) \), and the countable remainder of \( S(P) \) with empty ordering; it is obvious that if \( \varphi \) belongs to the first set and \( \psi \) to the second, then \( \varphi < \text{Th}(\mathcal{M}) \psi \) holds.

Suppose that \( \mathcal{M} \) and \( \mathcal{N} \) are models for \( P \) and \( \mathcal{M} \subset \mathcal{N} \). Since \( \varphi < \psi \) is a \( \sum_1 \) sentence, \( \varphi < \text{Th}(\mathcal{N}) \psi \) implies \( \varphi < \text{Th}(\mathcal{M}) \psi \), i.e., \( \varphi < \text{Th}(\mathcal{N}) \subset \text{Th}(\mathcal{M}) \). Also, \( \neg T \subset \text{Th}(\mathcal{M}) \) holds for any \( T \supset P \) and \( \mathcal{M} \models T \). The next proposition describes models for \( P \) satisfying \( \varphi < \text{Th}(\mathcal{M}) \).

**Proposition 1.** Let \( \mathcal{M} \) be a model for \( P \); then \( \varphi < \text{Th}(\mathcal{N}) \) iff \( \mathcal{N} \prec \sum_1 \mathcal{M} \).

**Proof.** Since \( \varphi < \text{Th}(\mathcal{N}) \), it immediately follows \( \varphi < \text{Th}(\mathcal{M}) \) if \( \mathcal{M} \models \varphi \). Conversely, suppose that \( \varphi < \text{Th}(\mathcal{M}) \), and let \( \varphi \) be a \( \sum_1 \) sentence true in \( \mathcal{M} \). Since for \( \sum_1 \) sentences \( \varphi \), \( \varphi < \text{Th}(\mathcal{N}) \psi \) holds (see 5.3 in [SM]), we get that \( \mathcal{M} \models \text{Thm}_P(\varphi \gamma) \), then, for every other sentence \( \psi \in S(P) \), either \( \varphi < \text{Th}(\mathcal{M}) \psi \) or \( \psi < \text{Th}(\mathcal{M}) \varphi \) holds. This and the assumption \( \varphi < \text{Th}(\mathcal{M}) \psi \) imply that \( \varphi \) belongs to the linearly ordered part of \( \varphi \). Thus, \( \varphi \) is a theorem of \( P \) and consequently \( \mathcal{M} \models \varphi \). Since for all \( \sum_1 \) sentences \( \varphi \), \( \varphi \models \varphi \), we get \( \mathcal{N} \prec \sum_1 \mathcal{M} \).

Using Corollary 2.9.1 from [MI], asserting that \( \mathcal{N} \prec \sum_1 \mathcal{M} \), \( \mathcal{N} \models \varphi \), \( \mathcal{M} \cong \mathcal{M} \) = \( n \), we get the following algebraic characterization of models satisfying \( \varphi < \text{Th}(\mathcal{M}) \).

**Corollary 1.** \( \varphi < \text{Th}(\mathcal{M}) \) iff \( \mathcal{M} \models \varphi \). We use the following lemma to prove that, although the ordering \( \varphi < \text{Th}(\mathcal{M}) \) is linear for any model \( \mathcal{M} \) of the theory \( P + \neg \text{Con}_P \), the ordering \( \varphi < \text{Th}(\mathcal{M}) \) is not linear.

**Lemma 2.** There is a sentence \( \varphi \), independent of the theory \( P + \neg \text{Con}_P \), such that \( P \models \varphi \leftrightarrow ((\varphi \rightarrow \text{Con}_P) < (\neg \text{Con}_P \rightarrow \varphi)) \).

**Proof.** Gödel’s diagonalization technique and usual arguments for sentences of the Rosser type.

**Proposition 2.** There are two sentences \( \sigma, \psi \in S(P) \), such that neither \( \sigma < \neg \text{Con}_P \psi \) nor \( \psi < \neg \text{Con}_P \sigma \) holds.

**Proof.** Let \( \sigma \) be the sentence \( \varphi \rightarrow \text{Con}_P \) and \( \psi \) the sentence \( \neg \text{Con}_P \rightarrow \varphi \), where \( \varphi \) is the sentence from Lemma 2. From the same lemma we get that there are two models \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) of the theory \( P + \neg \text{Con}_P \) such that \( \mathcal{M}_1 \models \varphi \) and \( \mathcal{M}_1 \models \neg \varphi \). It is obvious that \( \sigma < \text{Th}(\mathcal{M}_1) \psi \) and \( \neg(\sigma < \text{Th}(\mathcal{M}_2) \psi) \). But \( \mathcal{M}_2 \models \neg \text{Con}_P \) implies that the ordering \( \varphi < \text{Th}(\mathcal{M}_1) \sigma \) is linear; so we get \( \psi < \text{Th}(\mathcal{M}_2) \sigma \). Thus, neither \( \sigma < P + \neg \text{Con}_P \psi \) nor \( \psi < P + \neg \text{Con}_P \sigma \) holds.
Models $M_1$ and $M_2$ described in the proof of Proposition 2 don’t have a common extension which is a model for $P$; $M_1 \subset \mathfrak{M}$, $M_2 \subset \mathfrak{M}$ and $\mathfrak{M} \models P$ would imply $\sigma <_{\text{Th}(\mathfrak{M})} \psi$ and $\psi <_{\text{Th}(\mathfrak{M})} \sigma$, which is a contradiction. Thus, we get the following corollary.

**Corollary 2.** Peano arithmetic does not have the Joint Embedding Property, i.e. there are two models of $P$ which cannot be embedded in any common extension which is a model for $P$. □

Let $T$ be a consistent extension of $P$; if the theory $T' = T + \neg \text{Con}_P$ is consistent and $\mathfrak{M} \models T'$, then the ordering $<_{\text{Th}(\mathfrak{M})}$ is a linear extension of the ordering $<_T$. Using the following lemma [KR] we prove that the ordering $<_T$ induced by any consistent extension $T$ of $P$ can be extended in a similar way.

**Lemma 3** (Kreisel). *The theory $P + \neg \text{Con}_P$ is a $\Pi_1$-conservative extension of $P$, i.e. for any $\Pi_1$-sentence $\varphi$, $P + \neg \text{Con}_P \vdash \varphi$ iff $P \vdash \varphi$.*

**Proposition 3.** For any consistent theory $T \supset P$ there is a theory $T' \supset P$ such that the ordering $<_T$ is a linear extension of $<_T$.

*Proof. Let $S = \{ \varphi < \psi \mid \varphi <_T \psi; \varphi, \psi \in S(P) \}$. The theory $P + \neg \text{Con}_P + S$ is consistent; for any finite $S_0 \subset S$, Lemma 3 and $P + \neg \text{Con}_P \vdash \neg \land_{\varphi \in S_0}$ would imply $P \vdash \neg \land_{\varphi \in S_0}$; this is a contradiction, since $T \supset P$ is consistent and $T \vdash \land_{\varphi \in S_0}$. Let $\mathfrak{M}$ be a model for the theory $P + \neg \text{Con}_P + S$; the ordering $<_{\text{Th}(\mathfrak{M})}$ is a linear extension of $<_T$ induced by the consistent extension $T' = \text{Th}(\mathfrak{M})$ of $P$.*

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Prirodno-matematički fakultet

34000 Kragujevac, Jugoslovija

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