A TREE AXIOM\textsuperscript{1}

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\textbf{Abstract.} In connection with my previous results from 1935, and results of other mathematicians (Tarski, Erdős, Hajö, Keisier, Baumgartner...) the following Tree (or Dendrity) Axiom is formulated: For any regular uncountable ordinal \( n \) there exists a tree \( A_n \) of height (rank) \( n \) such that \( |X| < |n| \) for every level \( X \) as well as for every subchain \( X \) of \( A_n \). In other words, the following assertion \( D_n \) holds: There exists a tree \( T \) such that for every regular ordinal \( n > \omega_0 \) the conditions (2:0), (2:1), (2:2) hold.

\textbf{0.} Studying since 1932 the well-known Suslin problem concerning simple ordered sets (chains), and transforming this problem into a problem concerning trees or ramified tables, I was lead to consider trees \( (T, \leq) \) of height or rank \( \gamma T = \omega_1 \) such that \( |X| < \aleph_1 \), where \( X \) runs through the set of all rows of \( (T, \leq \) and the system of all subchains of \( (T, \leq) \). I studied such trees irrespectively of their existence (cf. Kunen \( 1980, \ p. \ 69^{\text{th}-11} \)): “Suslin trees were introduced by Kurepa (see, e.g. Kurepa \( 1936 \)), who showed that there is an \( \omega_1 \)-Suslin tree iff there is a Suslin line (see Theorem 5.13)”, and Todorčević \( 1984, \ p. \ 246\_10 \): “Aronszajn, Suslin and Kurepa trees were introduced by Kurepa \( 1935, \ 1937 \) a and \( 1942 \)” (i.e. in Kurepa \( 1935 \) b, c, \( 1937 \) b and \( 1942 \) a of the present bibliography). I considered analogous situations for regular cardinals \( \omega_\beta \) for every ordinal \( \beta > 0 \), in particular for any inaccessible ordinal \( \beta > \omega_0 \).

After many years let me announce the following

\textbf{1.} Tree (or Dendrity) Axiom: For any regular uncountable ordinal \( n \) there exists a tree \( A_n \) of height or rank \( n \) such that \( |X| < |n| \); here \( X \) stands for any level or row of \( A_n \) or any subchain of the tree \( A_n \).

\textbf{2.} Statement \( D_n \): Let \( D_n \) or \( D(n) \) denote for any ordinal \( n \) the following assertion:

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\textsuperscript{2}Any infinite ordinal \( \alpha \) is said to be inaccessible if it is a limit regular, thus \( \text{cf} \, \alpha = \omega_\alpha \) and \( \text{cf} \, \omega_0 = \alpha \).
There exists a tree $T$ such that

\begin{enumerate}
\item $\gamma T = n$,
\item $|R_\alpha T| < |n|$ for any $\alpha < \gamma T$
\item $|L| < |n|$ for any chain $L$ in $T$.
\end{enumerate}

2.3. Notation. For a tree $T$ and every ordinal $\alpha$ we denoted by $R_\alpha T$ the set of all elements $x$ of $T$ such that the corresponding left interval $T(\gamma, x) := \{ y \mid y \in T, y < x \}$ has the order-type $\alpha$. The first $\alpha$ such that $R_\alpha T = \nu$ (vacuous or empty set) was called the rank or height of $T$ and denoted by $\gamma T$; $R_\alpha T$ was called the $\alpha$-th level (row) of the tree $T$. In this way we obtained the well-determined fundamental disjoint partition $T = \bigcup R_\alpha T$ ($\alpha < \gamma T$) of every tree $T$.

2.4. The pseudorank $\gamma T$ of $T$ is the greatest limit ordinal $\leq \gamma T$; if $\gamma T < \omega_1$, then $\gamma T := 0$.

2.5. A classification of trees $T$. For any $\alpha < \gamma T$ let $m_\alpha T := |R_\alpha T|, m T := \sup m_\alpha T$.

\begin{enumerate}
\item $T$ is said to be large if for some $\alpha < \gamma T$ one has $m_\alpha T \geq |\text{cf} \gamma T|$; $T$ is said to be narrow if $m T < |\text{cf} \gamma T|$ and moreover, if $\text{cf} \gamma T = \omega + 1$ then $m T < \omega_\beta$. If $T$ is neither narrow nor large, $T$ is said to be ambiguous, nice or idoneous.
\end{enumerate}

2.6. The falsity of $D\omega$ is the content of König’s [1927] Infinity Lemma. Independently of this lemma I proved the falsity of $D\omega_\nu$ for any $\omega_\nu$ cofinal with $\omega_1$ – a particular case of the fact that every infinite narrow tree is equinumerous with one of its own subchains (v. These [1933] p. 80 Th. 3\textsuperscript{bis}).

3. Genesis. In 1934 I gave the definition of decreasing trees of sets, $T$ with an erroneous statement that $T$ contains a chain intersecting every row of $T$ (lapis calami; it was not indicated that $T$ should be “narrow”).

3.1. In the next note [1934 d] the error was notified and it was indicated that Aronszajn gave me an example of an ambiguous $\omega_1$-tree having no $\omega_1$-branche. Aronszajn’s construction was published in Kurepa [1935 b, c, p. 96]\textsuperscript{5}.

3.2. At the same place was published my construction (found in 1934 after that of Aronszajn) and based uniquely on order considerations concerning the ordered set $(Q, \leq)$ of rational numbers. My starting point was the tree $\sigma(Q, \leq) := \sigma_0$ of all well-ordered bounded subsets of $(Q, \leq)$ ordered by the relation “to be an initial segment of”; $\sigma_0$ is a tree of nonattained rank $\omega_1$ and its levels $R_\alpha \sigma_0$ are of power $\aleph_0$ for $\alpha < \omega$ and $2\aleph_0$ for $\omega \leq \alpha < \omega_1$.

3.3. I indicated that instead of $(Q, \leq)$ one could consider the Hausdorff set $H_0 := (l + m + n)^{\omega + 1}$, the system of all $\omega$-sequences $f$ of numbers of a given ordered set \{l < m < n\}, such that a right section of $f$ equals the constant $\omega_0$-sequence $m, m, \ldots$; the set $H_0$ is ordered by the principle of first differences. At p. 9712–10 I indicated: “Comme on a construit $\sigma_0$, $S_0$ à partir de $H_0$, on construit, à partir de $H_\beta$, les suites ramifiées $\sigma_\beta$, $S_\beta$. Nous ne le ferons pas”. Thus $\sigma_\beta := \sigma(H_\beta, \leq), S_\beta$ is a $\omega_{\beta+1}$-tree $\subset \sigma_\beta$ of breadth $|\omega_\beta|$.

\textsuperscript{5}I do not know why Aronszajn did not publish his construction and I am sorry that he didn’t. Aronszajn and I met in Paris quite frequently in the years 1933–1935 and in 1937 (we prepared both our Thèses with Fréchet). In particular he was a witness of the writing of my Thèse; he had a copy of the manuscript of my Thèse before its publication.
3:4. As a matter of fact the construction of $S_0$ was transferable *verbatim* for construction of $S_\beta \subset \sigma_\beta$ provided the ordinal $\beta$ be such that $\sum_{\xi<\omega_\beta} 3^{[\xi]} < 3^{|\omega_\beta|}$; this situation occurs for every regular ordinal $\beta$, provided GCH is accepted.

3:5. Independently of GCH the existence of $S_\beta$, i.e. of ambiguous $A(\omega_{\beta+1})$-trees, was proved in Kurepa [1968 b] for every ordinal $\beta$; the construction is based on the following result, important by itself.

**Theorem.** Kurepa [1935 b, c Th. 6 p. 89]. *Let $T$ be a tree such that for every element $t$ of $T$ the set $R_0(t, \cdot)$ of all immediate successors of $t$ in $T$ is infinite; let this set be ordered totally in such a way that the ordering has no minimal element. We consider the corresponding natural ordering of $(T, \leq)$. Let $a$ be any ordinal limit between 0 and $\gamma_T$. The sets $(x, a)_T := \cup_{\xi<\alpha} R_\xi T$ and $R_\alpha T$ are mutually dense (i.e. everywhere dense one into another) in this natural ordering $\leq n$ if and only if $\gamma[x]_{(T, \leq)} = \gamma(T, \leq)$ for every $x \in T$; one takes $[x]_T := \{ y \in T, y$ is comparable to $x \}$.*


3:6:1. In Kurepa [1935 b, c p. 100-13] one reads with corresponding italic characters: "Nous ne savons pas s’il existe une suite distinguée dont le rang serait un nombre inaccessible. Au contraire, quel que soit l’ordinal $\beta$, on peut démontrer l’existence d’une suite distinguée de rang $\omega_{\beta+1}$. On rencontrera à plusieurs reprises des suites distinguées". In the present terminology, we do not know whether there exists a tree $D(\omega_{\xi})$ for inaccessible $\omega_{\xi}$. On the contrary, for every ordinal $\beta$ one can prove the existence of a $D(\omega_{\beta+1})$-tree. We shall encounter such trees a lot of times.

3:6:2. In Kurepa [1968:2, p. 153-1] one reads “The problem of the existence of $A_\nu$ for inaccessible $\nu > \omega_0$ remains open”, $A_\nu$ denotes any ambiguous $\omega_{\nu}$-tree having no $\omega_{\nu}$-branche.

To make the terminology precise: every regular limit cardinal (ordinal) is said to be inaccessible; $k$ is said to be strongly inaccessible if $k$ is inaccessible and moreover for every $x < k$ one has $2^{[x]} < k$.

3:6:3. According to Erdös-Tarski [1961] $D(k) \iff k \rightarrow (k)^2$ for every strongly inaccessible $[s, i]$ cardinal $k$.

3:6:4. According to Hanf [1964], for many s.i. cardinals one has $D(k)$. In particular, for the first s.i. ordinal $i$ one has $D(i)$.

3:6:5. Now, we formulate that $D(k)$ holds for every inaccessible ordinal $k > \omega_0$ as well as for every $k = \omega_{\beta+1}$; irrespectively of whether the ordinal $\beta$ is regular or singular; in particular our axiom yields $A_{\omega_{\beta+1}}$.


4:1. What about maximum antichains in trees? In the period until 1937 it was not known whether the ambiguous or idoneous $\omega_1$-trees, constructed in 1934 first by Aronszajn, and then by myself, contain an uncountable antichain; this question was indicated in my Thèse as an open problem (see Thèse, Introduction p. 34-13,
p. 1347–71); the problem was solved in 1937 affirmatively: from Glina I sent in 193708:23:1 a manuscript to Banach St. (Leopol) a solution of the problem. The paper was received in 193708:31 in Studia Mathematice and published in 1940; I read no proof-sheets; I received the reprints in 194902:19 in Zagreb. The paper [1940] contains no typographical error; only, p. 256,13 non \( \rightarrow \) disjoints non.

4:2. The existence of \( A(\omega) \) trees with no \( \aleph_1 \)-antichain was considered by me as a postulate (see Thèse 1935 p. 134); this fact was confirmed as late as 1967–1971 by Jech, Tennenbaum, and Solovay.

5. Occurring of inaccessible numbers

5:0. In the Tree Axiom the role of inaccessible numbers is particularly important, and therefore the Tree Axiom can be considered mainly as a large cardinal axiom.

In this connection it is instructive to indicate the following facts.

5:1. In my doctoral dissertation the main alternative for infinite cardinals was the distinction between maximum and supremum and the question whether in given circumstances one has really a maximum and not only a supremum (under this main idea was formulated my ramification hypothesis). And in this respect the case of inaccessible numbers had a crucial role.

5:2. In my Thèse (p. 109) for any tree \( T \) I introduced a cardinal number \( b'T \) which in the particular case when \( T \) is a decreasing tree of sets is just the supremum of \( |D|, D \) running through the system of all disjoint subsystems of \( T' := \{X, X \in T \) or \( X = Y \setminus Z, where Y, Z \in T and Y \supset Z\}. Then I proved the following

5:2:1. Theorem [Thèse p. 110, Théorème 3]. Unless the tree \( T \) is of inaccessible rank, the supremum \( b'T \) is attained.

This has an obvious consequence concerning ordered chains \( (E, \leq) \): Unless cf \( p_2E \) is inaccessible, the ordered space \( (E, \leq) \) contains a disjoint system of open intervals of cardinality \( p_2E := \) the cellularity of \( (E, \leq) \), i.e. sup \( |D|, D \) being a disjoint system of open sets in \( (E, \leq) \).

5:2:2. This fact is transferable to topological spaces, as was published without quotation of my result, in Erdős-Tarski [1943]. In this respect it is instructive to quote the starting lines of this paper (p. 315^4–10^): “In this paper we shall be concerned with a certain particular problem for the general theory of sets, namely with the problem of the existence of families of mutually exclusive sets with a maximal power. It will turn out – in a rather unexpected way – that the solution of these problems essentially involves the notion of the so-called “inaccessible numbers”. In this connection we shall make some general remarks regarding inaccessible numbers in the last section of our paper”.

REFERENCES


A Tree Axiom

11


Notice: Ligne 135: remplacer par ceci: remarquable dû à moi et dont la démonstration correcte sur ma demande, est due à Aronszajn; je lui avais donné mon manuscrit (démonstration), que, à Warsawa, j’avais donné, personnellement, pour Fundamenta Mathematicae, à Sierpiński, avant mon départ de Warsawa pour Paris (v. Introduction n° 4).

Problèmes.

Étudiez une suite ε de familles, au plus énumérables, $F_2(\mathbb{E})$, dont chacune soit composée d’ensembles linéaires formés deux à deux disjointes et vérifiant les conditions 1° et 2° que voici: 1°) pour tout $\alpha, \beta, \gamma \in \mathbb{E}$, les relations $A \subseteq B$, $B \subseteq C$ entraînent $A \subseteq C$; 2°) pour tout $\alpha, \beta, \gamma \in \mathbb{E}$, on a bien $AB = \emptyset$, et pour tout $\alpha, \beta, \gamma \in \mathbb{E}$, on a bien $\emptyset \subseteq AB$. (En relation est prise au sens strict).

Problèmes de K. Kurepa.

This text was written in 1937 by Sierpiński.

4At this opportunity, Sierpiński gave me back the formulation of my problem written in March 1937 by his own hand; I gave him my manuscript (positive answer to my problem), he gave me back the problem; v. Facsimile above.

Remark. The last phrase in the review in Zbl 20 (1940) 108 written by Blumberg (Columbia) should read: “This proposition has been announced by the author, who personally, before leaving Warsawa for Paris, end of April 1937, gave it to Sierpiński to publish in Fundamenta Mathematicae: in Paris the author discovered a gap in his proof; then he asked Aronszajn to bridge the gap; Aronszajn succeeded and found the proof displayed in the present paper (cf. Introduction, n° 4).”


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