ON THE ABSOLUTE SUMMABILITY OF LACUNARY FOURIER SERIES

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Abstract. Let $f \in L[-\pi, \pi]$ and let its Fourier Series $\sigma(f)$ be lacunary. The absolute convergence of $\sigma(f)$ when $f$ satisfies Lipschitz condition of order $\alpha$, $0 < \alpha < 1$, only at a point and when $\{n_k\}$ satisfies the gap condition $n_{k+1} - n_k \geq A n_k^\beta k^\gamma$ $(0 < \beta < 1$, $\gamma \geq 0)$ is obtained by Patadia and Shah when $\alpha\beta + \alpha\gamma > (1 - \beta)/2$. Here we study the absolute summability of $\sigma(f)$ when $\alpha\beta + \alpha\gamma \leq (1 - \beta)/2$.

1. Let
\[
\sum_{k=1}^{\infty} (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x)
\]
be the Fourier series of a $2\pi$-periodic function $f \in L[-\pi, \pi]$ with an infinity of gaps $(n_k, n_{k+1})$, where $\{n_k\}$ $(k \in N)$ is a strictly increasing sequence of natural numbers. Noble [7], Kennedy [4, 5, 6], and several other mathematicians, have studied the absolute convergence of the Fourier series (1.1), as well as the order of magnitude of Fourier coefficients, by considering various properties of $f$ either on an arbitrary subinterval or on an arbitrary subset of $[-\pi, \pi]$ of positive measure. This way they obtained a number of results under different lacunary conditions. Izumi and Izumi [3], Chao [1], and Patadia and Shah [8], have studied this problem for the Fourier series (1.1) with some lacunae when the function satisfies Lipschitz condition only at a point. Chao [1] proved the following theorems:

Theorem A. [1; Theorem 1]. If
\[
\begin{align*}
(i) & \quad f \in \text{Lip} \alpha \,(\alpha > 0) \text{ at a point } x_0 \in (-\pi, \pi), \\
(ii) & \quad n_{k+1} - n_k \geq A F(n_k)
\end{align*}
\]
where $F(n_k) \uparrow \infty$ as $k \to \infty$, $F(n_k) \leq n_k$ for all $k$ and $A$ is a positive constant, then
\[
a_{n_k}, b_{n_k} = O(F(n_k)^{-\alpha}), \; k = 1, 2, \ldots
\]

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\textbf{Theorem B.} [1; Theorem 2]. If $f$ satisfies (1.2) and if
\[ n_{k+1} - n_k \geq A \beta^k \kappa \quad (0 < \beta < 1, \ \gamma \geq 0) \]  
(1.5)
where $A$ is a positive constant, then the Fourier series (1.1) of $f$ converges absolutely when $\alpha \beta + \alpha \gamma + \beta > 1$

Furthermore, Patadia and Shah [8] considered the same gap condition (1.5) and proved the following theorem:

\textbf{Theorem C.} If $f$ satisfies (1.2), and if \{\textit{n}_k\} satisfies (1.5), then
\[ \sum_{k=1}^{\infty} \left( |a_{n_k}|^r + |b_{n_k}|^r \right) < \infty \quad 0 < r \leq 1 \]  
(1.6)
when $\alpha \beta r + \alpha \gamma > (1 - r/2)(1 - \beta)$.

We observe that the particular case of theorem C when $r = 1$ provides us with a generalization of Theorem B, ensuring the absolute convergence of the Fourier series (1.1) when $\alpha \beta + \alpha \gamma > (1 - \beta)/2$. It may be noted here that when $\alpha \beta + \alpha \gamma = (1 - \beta)/2$, the absolute convergence of (1.1) is obtained by Patadia and Shah [9] by taking at a point a little stronger condition than Lip $\alpha$ on $f$.

Now, it is quite natural to inquire into the behaviour of the Fourier series (1.1) of a function $f$ in Lip $\alpha$ at a point, when $\alpha \beta + \alpha \gamma \leq (1 - \beta)/2$. In this regard, we propose to study the absolute summability $(c, \theta)$ of the series (1.1). We prove the following theorem:

\textbf{Theorem.} If $f \in$ Lip $\alpha(0 < \alpha < 1)$ at a point $x_0 \in (-\pi, \pi)$, and if \{\textit{n}_k\} satisfies (1.5) with some suitable constant $A$, then the Fourier series (1.1) of $f$ is absolutely summable $(c, \theta)$ for $0 < \theta \leq 1$ when
\[ \alpha > \max \left\{ \frac{1 - \beta - \theta - \gamma \theta}{\beta + \gamma}, \frac{2 - 3(\beta - \gamma) + \beta \theta - \theta}{\beta + \beta \gamma} \right\} \]

\textit{Remark 1.} Theorems 1 and 2 due to Patel [10] are particular cases of this theorem when $\theta = 1, \ \gamma = 0, \ \text{and} \ \theta = 1/2, \ \gamma = 0$ respectively.

\textit{Remark 2.} It is interesting to observe that when $\gamma = 1$, the theorem gives the absolute summability $(c, 1)$ of the Fourier series (1.1) for every $\alpha > 0$; and that, when $\gamma = 3/2$, we get the absolute summability $(c, 1/2)$ of (1.1) for every $\alpha > 0$.

2. We need the following lemma due to Patadia and Shah [9].

\textbf{Lemma.} If \{\textit{n}_k\} satisfies (1.5) with $A > 2^M - 1$, $M$ being a positive integer greater than, $\delta$, where $\delta = (1 + \gamma)/(1 - \beta)$, then
\[ n_k \geq k^\delta \quad \text{for all} \ k \in N \]  
(2.1)
Proof of the Theorem. For a real number $s$, which is not a negative integer, put $E_n^s = \left( \frac{n^s}{n} \right)$ where $n \in \mathbb{N}$ and $E_n^0 = 1$. Denoting the $n$-th Cesaro mean of order $\theta > 0$ by $\sigma_n^\theta(x)$, and replacing the absent terms in (1.1) by zeros, we have \cite{2}:

$$
|\sigma_n^\theta(x) - \sigma_{n_k-1}^\theta(x)| = \frac{1}{n_k \cdot E_{n_k}^\theta} \left| \sum_{p=1}^{k} E_{n_k-n_p}^{\theta-1} \cdot n_p \cdot (a_{n_p} \cos n_p x + b_{n_p} \sin n_p x) \right|
$$

\[
\leq \frac{1}{n_k \cdot E_{n_k}^\theta} \left\{ \left| n_k (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x) \right| + \left| \sum_{p=1}^{k-1} E_{n_k-n_p}^{\theta-1} \cdot n_p \cdot (a_{n_p} \cos n_p x + b_{n_p} \sin n_p x) \right| \right\}. \quad (2.2)
\]

Let $0 < \theta \leq 1$. Now,

(i) $E_n^\theta \approx \frac{n^\theta}{\Gamma(\theta + 1)}$,

(ii) $a_{n_k}, \ b_{n_k} = O\left(\frac{1}{n_k^{\alpha/\beta} \cdot k^{\gamma/\alpha}}\right)$, $k = 1, 2, 3, \ldots$,

by taking $F(n_k) = n_k^{\beta/\gamma} k^\gamma$ in Theorem A, and

(iii) $|n_k - n_p| \geq |n_k - n_{k-1}|$ for $p = 1, 2, 3, \ldots, k - 1$

\[
\geq A n_k^{\beta/\gamma}, \quad \text{by (1.5)}.
\]

Hence, from (2.1) and (2.2), we obtain

$$
|\sigma_n^\theta(x) - \sigma_{n_k-1}^\theta(x)|
$$

\[
= \text{o}(1) \left\{ n_k^{\alpha_{\gamma/\alpha}} k^{-\gamma/\alpha} + \sum_{p=1}^{k-1} \frac{1}{(n_k - n_p)^{1-\theta}} \cdot n_p \cdot n_p^{\alpha_{\gamma/\alpha}} p^{-\gamma/\alpha} \right\}
\]

\[
= \text{o}(1) \left\{ n_k^{1-\alpha/\beta} k^{-\gamma/\alpha} + \frac{1}{n_k^{\beta/\gamma} k^{\gamma}} \sum_{p=1}^{k-1} n_p^{1-\alpha_{\gamma/\alpha}} \cdot p^{-\gamma/\alpha} \right\}
\]

\[
= \text{o}(1) \left\{ n_k^{1-\alpha/\beta} k^{-\gamma/\alpha} + \frac{1}{n_k^{\beta/\gamma} k^{\gamma}} \cdot k \cdot n_k^{1-\alpha_{\gamma/\alpha}} \right\},
\]

as $p^{-\alpha/\gamma} \leq 1$ and $n_p^{1-\alpha_{\gamma/\alpha}} \leq n_k^{1-\alpha_{\gamma/\alpha}}$, $0 < \alpha, \beta < 1$. Therefore

$$
|\sigma_n^\theta(x) - \sigma_{n_k-1}^\theta(x)| = \text{o}(1) \left\{ \frac{1}{n_k^{\theta+\alpha_{\gamma/\alpha}}} + \frac{1}{n_k^{\theta+\beta+\gamma/\alpha}} + \frac{1}{n_k^{\theta+\beta-\alpha/\gamma}} + \frac{1}{n_k^{\theta+\beta-\alpha_{\gamma/\alpha}+\gamma/\alpha}} \right\}
$$

\[
= \text{o}(1) \left\{ k^{\delta(\theta+\alpha_{\gamma/\alpha}+\gamma/\alpha)} + \frac{1}{k^{\delta(\theta+\beta-\alpha/\gamma)+\gamma/\alpha}} \right\}
\]

\[
= \text{o}(1) \left\{ \exp_k \left( \frac{\theta + \alpha_{\gamma/\alpha} + \gamma/\alpha + 1}{1 - \beta} \right) \right\} + \exp_k \left( \frac{\theta + 2\beta - \alpha/\gamma + \alpha_{\gamma/\alpha} + \gamma/\alpha - 1}{1 - \beta} \right), \quad (2.3)
\]
as \( \delta = (1+\gamma)/(1-\beta) \); \( \exp_k A \) denotes \( k^{-A} \). Finally, since \( \alpha > (1-\beta - \theta \gamma)/(\beta + \gamma) \), it follows that \( (\theta + \alpha \beta + \gamma \theta + \alpha \gamma)/(1 - \beta) > 1 \); and since

\[
\alpha > \frac{2-3\beta - \gamma + \beta \theta - \theta}{\beta + \beta \gamma}
\]

we have

\[
\frac{\theta + 2\beta - \beta \theta + \alpha \beta + \alpha \beta \gamma + \gamma - 1}{1 - \beta} > 1.
\]

Hence, from (2.3) we have

\[
\sum_{k=1}^{\infty} |\sigma_{n_k}^\alpha (x) - \sigma_{n_{k-1}}^\alpha (x)| < \infty,
\]

which implies the absolute summability \((c, \theta)\) of (1.1). This completes the proof of the theorem.

REFERENCES


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