A NIL-EXTENSION OF A COMPLETELY SIMPLE SEMIGROUP

Stojan Bogdanović and Svetozar Milić

Abstract. We describe semigroups which are nil-extensions of completely simple semigroups and in particular nil-extensions of left groups or rectangular bands.

In this paper we consider power regular semigroups in which idempotents are primitive. These semigroups are nil-extensions of a completely simple semigroups (Theorem 1).

Power regular semigroups are considered in [1]. A semigroup $S$ is power regular if for every $a \in S$ there exists $m \in N$ such that $a^m \in a^mSa^m$. A semigroup $S$ is power completely regular if for every $a \in S$ there exist $x \in S$ and $m \in N$ such that $a^m = a^mxa^m$, $a^mz = xa^m$.

If $e$, $f$ are idempotents of a semigroup $S$, we shall write $e \leq f$ if $ef = fe = e$. An idempotent is called primitive if it is nonzero and is minimal in the set of non-zero idempotents relative to the partial order $\leq$. By nil-extension we mean an ideal extension by a nil-semigroup. A semigroup $S$ with zero 0 is a nil-semigroup if for every $a \in S$ there exists $n \in N$ such that $a^n = 0$. By $E$ denote the set of all idempotents of a semigroup.

For undefined notions and notations we refer to [2], [4] and [7].

Lemma 1. If $S$ is power regular semigroup all of whose idempotents are primitive, then $S$ is power completely regular with maximal subgroups given by $G_e = eSe$ ($e \in E$).

Proof. For $a \in S$ there exist $x \in S$ and $m \in N$ such that $a^m = a^mxa^m$. For $a^k \in S$, where $k > m$, there exist $y \in S$ and $n \in N$ such that $a^k = a^kya^k$. Assume that $e = a^m$ an $f = a^nya^m$. Then

$$f^2 = a^kya^mxa^kya^m = a^kya^m(a^mxa^m)a^{k-m}ya^m = a^kya^m(a^k-m)a^{k-m}ya^m = a^kya^m(a^k-m)a^{k-m}ya^m$$

$$= a^kya^mxa^m = a^kya^m = f$$

$$ef = a^mxa^kya^m = a^mxa^m(a^k-m)a^{k-m}ya^m = a^mxa^m$$

$$fe = a^kya^mxa^m = a^kya^m = f.$$
Hence, $ef = fe = f$. From this it follows that
\[ a^m = a^m xa^m = ea^m = fa^m = a^{kn}ya^m xa^m \in a^{m+1} Sa^m \]
i.e. $S$ is power completely regular [1, Proposition 3.2].

Let $e \in E$ and $u \in G_e$, then $u = eue \in eSe$ and thus $G_e \subseteq eSe$. Conversely, let $u \in eSe$, i.e. $uebe$ for some $b \in S$. Then $u^p \in G_f$ for some $p \in N$ and $f \in E$, so
\[ ef = eu^p(u^p)^{-1} = e(abe)^p(u^p)^{-1} = f \]
and dually $fe = f$. Hence, $e = f$. Therefore, $u^p \in G_e$. From this and Lemma 1 of [6] we have that $u^{p+1} \in G_e$, so
\[ e = u^{p+1}(u^{p+1})^{-1} = u \cdot u^p(u^{p+1})^{-1} = u^p(u^{p+1})^{-1} \cdot u \]
and since $eu = e(abe) = ebe = u = ue$ we have that $u \in G_e$ and therefore $eSe \subseteq G_e$.

**Lemma 2.** The unity $e$ of a minimal bi-ideal $B$ of $S$ is a primitive idempotent in $S$.

**Proof.** For an arbitrary idempotent $f \in S$, if $f = ef = fe$, then $f = ef e \in eSe \subseteq B$, so $e = f$ (since $B$ is a subgroup of $S$ [5, Lemma 2.6]).

**Lemma 3.** Let $K$ be the union of all minimal bi-ideals of $S$. Then $k$ is a completely simple kernel of $S$.

**Proof.** By Lemma 2.5 [5] $K$ is an ideal of $S$. By Lemma 2 we have that every idempotent from $K$ is primitive and since $K$ is a union of groups we have that $K$ is completely simple [4, Corollary III 3.6].

The following theorem is a generalization of a result of Munn [6, Theorem 2].

**Theorem 1.** The following conditions are equivalent on a semigroup $S$:

(i) $S$ is power regular and all idempotents of $S$ are primitive;

(ii) $S$ is a nil-extension of a completely simple semigroup;

(iii) ($\forall a, b \in S$) ($\exists m \in N$) ($a^m \in a^m b Sa^m$).

**Proof.** (i) $\Rightarrow$ (ii). By Lemma 1 we have that $S$ is power completely regular and maximal subgroups of $S$ are of the form $G_e = eSe$ ($e \in E$). Since $G_e(e \in E)$ is a minimal bi-ideal [5, Lemma 2.6], then by Lemma 3 we have that $S$ has a completely simple kernel $K$. It is clear that for every $a \in S$ there exists $m \in N$ such that $a^m \in K$.

(ii) $\Rightarrow$ (i). This implication follows immediately.

(ii) $\Rightarrow$ (iii). If $S$ is nil-extension of a completely simple semigroup, then for $a, b \in S$, $a^m$, $a^m ba^m \in G_e$ for some $m \in N$ (Lemma 1), so $a^m = a^m ba^m x$ for some $x \in G_e$. From this it follows that $a^m = a^m ba^m x(a^m)^{-1} a^m \in a^m b Sa^m$.

(iii) $\Rightarrow$ (ii). For $a = b$ we have that $a^m \in a^{m+1} Sa^m$, so by [1, Proposition 3.2] $S$ is power completely regular. Let $S$ have a proper ideal $I$. For $e \in E$ and
b \in I \text{ we have } e \in ebSe \subseteq 1. \text{ Hence, the intersection of all ideals of } S \text{ is nonempty, i.e. } S \text{ has a minimal ideal } K. \text{ Since } K \text{ is power completely regular we have that } K \text{ is completely simple (Theorem 2. [6]). For } a \in S \text{ and } b \in K \text{ we have that } \alpha^m \in a^m b Sa^m \subseteq K \text{ for some } m \in N.

\textbf{Theorem 2. The following conditions on a semigroup } S \text{ are equivalent:}

(i) \quad S \text{ is a nil-extension of a left group;}
(ii) \quad S \text{ is power regular and } E \text{ is a left zero band;}
(iii) \quad (\forall a, b \in S) \ (\exists m \in N) \ (a^m \in a^m Sa^m b).

\textbf{Proof. (i) } \Rightarrow (ii). \text{ This implication follows immediately.}

(ii) \Rightarrow (iii). \text{ By Theorem 1 we have that } S \text{ contains a completely simple kernel } K \text{ which is, in fact, a left group. For } a, b \in S \text{ there exist } m, n \in N \text{ such that } a^m, b^n \in K, \text{ so } a^m = x b^{n+1}, b^n = y a^m \text{ for some } x, y \in K. \text{ Since } a^m \in G_e \text{ for some } e \in E \text{ we have } a^m = a^m (a^m)^{-1} x b^{n+1} b = a^m (a^m)^{-1} x y a^m b \in a^m Sa^m b. \text{ (iii) } \Rightarrow (i). \text{ If the condition (iii) holds, then for } a \in S \text{ we have that } a^m \in a^m Sa^m a = a^m Sa^m b \text{ for some } m \in N \text{ and therefore by Proposition 3.2. [1] } S \text{ is power completely regular. For } e, f \in E \text{ we have that } f = f x f e \text{ for some } x \in S, \text{ so } f e = (f x f e) e = f, \text{ i.e. } E \text{ is a left zero band. Hence, } K \cup_{e \in E} G_e \text{ is a left group (see [2, Ex. e. §1.11].}

\textbf{Corollary 1. } S \text{ is a left group iff } (\forall a, b \in S) \ (a \in aS a).$

\textbf{Theorem 3. Let } S \text{ be a semigroup. If}

(\forall a \in S) (\exists x \in S) (\exists m \in N) (a^m = x a^{m+1}) \tag{1}

\text{then } S \text{ is a nil-extension of a left group.}

\textbf{Proof. Let (1) be satisfied in a semigroup } S. \text{ Then } a^m = x a^{m+1} = x^n a a^{m+1}. \text{ From this and from (1) it follows that}

x = x^2 a. \tag{2}

Furthermore, for } x \text{ there exist } y \in S \text{ and } n \in N \text{ such that } x^n = y x^{n+1} \text{ and }
y^2 = y x. \tag{3}

\text{From (2) and (3) it follows that}
y^2 = y y x = y^3 x^2 a = y^3 x x^2 a^2 = y^2 x^2 a^2 = y^2 x a = y a = y x a a^{m+1}. \text{ For } k = \max(m, n) \text{ we have}
y^2 = y x^{m+1} = y x^{k+1} a^{k+2} = y x^{n+1} x^{k-n} a^{k+2} = x^n a^{k-n} a^{k+2} = x^k a^{k+2} = x a^3, \text{ so } y = y^2 x = x a^3. \text{ Further,}
y^{m+2} = y^m y^2 = y^m y a = y^{m-1} y^2 a = y^{m-1} y^2 a = \ldots = y a^{m+1} = xa^3 xa^{m+1} = x a^3 a^m = a^{m+2}.
From this it follows that \( y^{m+2}z^{m+2} = a^{m+2}z^{m+2} \) and by (3) we have \( y = a^{m+2}z^{m+2} \). Hence
\[
a^m = xa^{m+1} = x^n a^{m+n} = yz a^{m+n} = a^{m+2}z^{m+1}x^{n+1}a^{m+n}
\]
so \( a^m \in a^{m+1}\alpha a^m \), i.e. \( S \) is power completely regular.

Let \( e, f \in E \). Then \( (ef)^m = x(ef)^{m+1} = xe(ef)^{m+1} \) for some \( x \in S \) and \( m \in N \). By uniqueness we have that \( x = x^2 = e \) and \( x = xe \). From this it follows that \( x = xe = xf \), so \( x = (ef)^m \). Furthermore, \( (ef)^m = (ef)^m = e = (ef)^m \) and
\[
(ef)^m = (ef)^m f = (ef)^{m+1} = (ef)(ef)^{m+1} = e(ef)^{m+1}.
\]
Therefore, \( ef = e \). So by Theorem 2 \( S \) is a nil-extension of a left group.

**DEFINITION 1.** \( S \) is a power group if \( S \) is a power regular with exactly one idempotent.

**THEOREM 4.** The following conditions are equivalent on a semigroup \( S \):

(i) \( S \) is a power group;

(ii) \( S \) is a nil-extension of a group;

(iii) \((\forall a, b \in S) (\exists m \in N) (a^m \in ba^m Sa^m)\)

**Proof.** (i) \( \Rightarrow \) (ii) This implication follows immediately.

(ii) \( \Rightarrow \) (iii) Let \( S \) be nil-extension of a group \( G \). For \( a, b \in S \) we have that 
\( a^m, a^mb, ba^m \in G \) for some \( m \in N \) and for each \( s \in S \), and then \( a^m = ba^msa^mbs \) for some \( x \in G \), i.e. \( a^m = ba^msa^mbs(a^m)\overline{m}a^m \in ba^msa^m\).

(iii) \( \Rightarrow \) (i) It is clear that \( S \) is power regular. We shall prove that \( S \) has only one idempotent. If \( e \) and \( f \) are idempotents from \( S \), then \( e = xf, f = ey \) for some \( x, y \in S \), so \( ef = xf = e, ef = ey = f \) thus \( e = f \).

**COROLLARY 2.** The following conditions are equivalent on a semigroup \( S \):

(i) \( S \) is a regular semigroup with only one idempotent;

(ii) \( S \) is a group;

(iii) \((\forall a, b \in S) (a \in baSa)\).

**REMARK.** (i) \( \Rightarrow \) (ii) is Corollary IV.3.6. of [4].

**LEMMA 4.** Let \( S \) be a semigroup. If
\[
(\forall a \in S)(\exists 1 x \in S)(\exists m \in N)(a^m = a^mx)^m
\]
then \( S \) is a power group.

**Proof.** Assume that (4) holds. Then for \( e, f \in E \) we have
\[
(ef)^m = (ef)^mg(ef)^m
\]
for some \( g \in S \) and \( m \in N \) and by uniqueness we have that
\[
g = g(ef)^m g
\]
It follows from \((ef)^m f g(ef)^m = (ef)^m\) that
\[ fg = g \] (7)
Similarly,
\[ ge = g. \] (8)
If \(m = 1\), then by (6), (7) and (8) we have that \(g = g^2\).
If \(m > 1\), then by (6), (7) and (8) we obtain \(g = g(ef)^m g = g(ef)^m-1 g\) and by uniqueness we have that
\[ (ef)^m = (fe)^m \] (9)
It follows from (5) and (9) that \((fe)^m-1 = (fe)^m-1 g(ef)^m-1 = (fe)^m-1 eg(ef)^m-1\), so
\[ eg = g. \] (10)
Similarly,
\[ gf = g. \] (11)
By (7), (8), (9) and (10) we have that \(g = g(ef)^m g = g^2\). Since \(g\) is an idempotent, then by uniqueness from (6) we obtain \(g = (ef)^m\). Hence,
\[ (ef)^{2m} = (ef)^m e(ef)^m = (ef)^m = (ef)^m f(ef)^m \]
and therefore \(e = f\). Thus \(S\) is a power group.

**Remark.** The converse of Lemma 4 is not true. For example, the semigroup \(S\) given by table 1

<table>
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<tbody>
<tr>
<td>a</td>
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<tr>
<td>b</td>
<td>b</td>
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<tr>
<td>c</td>
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is a power group. But, for \(c\) we have that \(c^2 = a \in G = \{a, b\}\) and there exist \(x = a\) and \(x = c\) such that \(c^2 = x^2c^2\).

It is easy to see that in the semigroup given by table 2 the condition (1) from Lemma 4 is satisfied.

**Theorem 5.** The condition (4) from Lemma 4 holds iff there is only one idempotent \(e\) in \(S\) and for every \(a \in S\) there exists \(m \in N\) such that \(a^m = a^m xa^m\), \(xe = x\).

**Proof.** If (4) holds, then by Lemma 4 \(S\) contains only one idempotent \(e\). By uniqueness we have that \(x = xa^m x\) and \(a^m x = e\) implies \(xe = x\).

Conversely, assume that for \(a \in S\) there exist \(x, y \in S\) and \(m \in N\) such that
\[ a^m = a^m xa^m = a^m ya^m. \] (12)
By uniqueness of the idempotent we have that \(a^m x = xa^m\). Hence, \(a^m\) is in a subgroup \(G_e\) of \(S\). By Lemma 1 [6] we have that \(xe = ex, ye = ey\) and \(xe, ye \in G_e\).
So by (12) we have that \( a^m exa^m = a^m eya^m \) and thus \( ex = ey \) by cancellation in \( G_e \). Hence, \( x = y \).

**Corollary 3.** [3] \( S \) is a group iff \( (\forall a \in S) (\exists x \in S) (a = axa) \).

**Theorem 6.** \( S \) is a nil-extension of a rectangular band iff

\[
(\forall a, b \in S)(\exists m \in N)(a^m = a^m ba^m).
\]

**Proof.** Let \( S \) be a nil-extension of a rectangular band \( E \). Then for \( a, b \in S \) there exists \( m \in N \) such that \( a^m = e \in E \) and by Lemma 1 we have that \( a^m ba^m = e \). Thus \( a^m = a^m ba^m \).

Conversely, it is clear that \( E \neq \emptyset \). For \( e, f \in E \) we have \( e = efe \) and \( f = fef \) and if \( ef = fe \), then \( e = ef = f \). Thus \( E \) is a rectangular band. For \( e \in E \) and \( x \in S \) we have that \( e = axe \), so \( ex \in E \), i.e. \( E \) is an ideal of \( S \) and clearly for every \( a \in S \) there exists \( m \in N \) such that \( a^m \in E \). Therefore, \( S \) is a nil-extension of a rectangular band.

**Corollary 4.** [4] \( S \) is a rectangular band iff \( (\forall a, b \in S) (a = aba) \).

**Corollary 5.** \( S \) is a nil-extension of a left zero band iff

\[
(\forall a, b \in S)(\exists m \in N)(a^m = a^m b).
\]

**Corollary 6.** \( S \) is a nil-semigroup iff \( (\forall a, b \in S)(\exists m \in N)(a^m ba^m = a^m b) \).

**References**