NOTE ON THE CIRCUITS OF A PERFECT MATROID DESIGN

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Abstract. For a perfect matroid design $M(E, r)$ on a finite set $E$ with $r$ as a rank function and $B \subseteq E$ a basis of $M(E, r)$, the number of circuits of cardinality $r(E) + 1$ containing $B$ is given.

Preliminaries. Throughout this paper we use some notions and results according to the standard literature on matroid theory (e.g., see [1, 2]). Let $E$ be a finite set and $M(E, r)$ a matroid on $E$ with $r$ as a rank function $r : \mathcal{P}(E) \rightarrow \mathbb{N}$, where $\mathbb{N}$ is the set of non-negative integers and $\mathcal{P}(E)$ the power set of $E$. A subset $S \subseteq E$ is called independent if $r(S) = |S|$, where $|S|$ denotes the cardinality of $S$, a basis of $M(E, r)$ being a maximal independent subset of $E$. A subset $S \subseteq E$ is called dependent if $r(S) \leq |S|$, a circuit of $M(E, r)$ being a minimal dependent subset of $E$. The span $S$ of a subset $S \subseteq E$ is

$$\tilde{S} = \{e \in E : r(S \cup \{e\}) = r(S)\}.$$  

For any integer $1 \leq k \leq r(E)$ we consider the set

$$CL[M(E, r), k] = \{S \subseteq E : S = \tilde{S}, \ r(S) = k\},$$

and $M(E, r)$ is called a perfect matroid design if every set of $CL[M(E, r), k]$ has a common cardinal $c(k), 1 \leq k \leq r(E)$. In the sequel we shall use without proofs (e.g., see [1, 2]) the following well-known results from matroid theory:

(a) $r(S) = r(\tilde{S})$ for each $S \subseteq E$,
(b) if $C$ is a circuit of $M(E, r)$, and $e \in C$, then $e \in C - \{e\}$,
(c) if $B$ is a basis of $M(E, r)$, then $\overline{B} = E$,
(d) if $C$ is a circuit of $M(E, r)$, then $r(C) = |C| - 1$,
(e) if $B$ is a basis of $M(E, r)$ and $e \in E - B$, then there exists a unique circuit $C(e, B)$ such that $e \in C(e, B) \subseteq B \cup \{e\}$.

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The main result. Throughout, $M(E, r)$ will be a perfect matroid design on $E$ and $B \subseteq E$ an arbitrary fixed basis of $M(E, r)$, the following sets of pairs being used:

$$A(B, k) = \{(F, e) : F \subseteq B, \ |F| = k, \ e \in \bar{F}\}, \text{ for any } 1 \leq k \leq r(E),$$

$$A(B) = \bigcup_{k=1}^{r(E)} A(B, k),$$

$$A(B, e) = \{(F, e) : (F, e) \in A(B)\}, \text{ for each } e \in E,$$

$$A(B, k, e) = \{(F, e) : (F, e) \in A(B), \ |F| = k\}, \text{ for each } e \in E \text{ and any } 1 \leq k \leq r(E).$$

Obviously, according to (a) and (c) we have

$$|A(B, k)| = \binom{r(E)}{k} c(k), \text{ for any } 1 \leq k \leq r(E). \quad (1)$$

Considering the function $\varepsilon : A(B) \rightarrow \{-1, 1\}$ defined by $\varepsilon[(F, e)] = (-1)^{r(E) - k}$, where $(F, e) \in A(B, k)$, we obtain from (1)

$$\sum_{(F, e) \in A(B)} \varepsilon[(F, e)] = \sum_{k=1}^{r(E)} (-1)^{r(E) - k} \binom{r(E)}{k} c(k). \quad (2)$$

For each $e \in E$ let $\alpha(e, B) = \sum_{(F, e) \in A(B, e)} \varepsilon[(F, e)].$

**Lemma 1.** If $e \in B$, then $\alpha(e, B) = 0$.

**Proof.** By (a), (c) and the definition of $A(B, k, e)$, if $e \in B$, then

$$|A(B, k, e)| = \binom{r(E) - 1}{k - 1}, \text{ for any } 1 \leq k \leq r(E).$$

Thus

$$\alpha(e, B) = \sum_{k=1}^{r(E)} (-1)^{r(E) - k} \binom{r(E) - 1}{k - 1} = (1 - 1)^{r(E) - 1} = 0.$$

**Lemma 2.** If $e \in E - B$, then $\alpha(e, B) \in \{0, 1\}$.

**Proof.** Let $C(e, B)$ be as in (e) and $|C(e, B)| = p$. Obviously, $p \leq r(E) + 1$ by (a), (c) and (e). Therefore, if $F \subseteq B$, then by (b) and (d) we have

$$(F, e) \in A(B) \Leftrightarrow C(e, B) \subseteq F \cup \{e\}.$$
that is,

\[ \alpha(e, B) = \sum_{k=p-1}^{r(E)} (-1)^{r(E)-k} \binom{r(E) - p + 1}{k - p + 1} \text{ or } \alpha(e, B) = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m}, \]

where \( n = r(E) - p + 1 \) and \( m = k - p + 1 \). Consequently

\[ \alpha(e, B) = \begin{cases} (1 - 1)^n = 0, & \text{if } n > 0, \\ 1, & \text{if } n = 0. \end{cases} \]

Remark. From the above lemmas it follows that \( \alpha(e, B) = 1 \) iff \( e \not\in B \) and \( |C(e, B)| = r(E) + 1 \), that is, iff \( B \cup \{ e \} \) is a circuit of \( M(E, r) \) by (d), (a) and (c).

Let us denote by \( \omega[B, r(E) + 1] \) the number of circuits of cardinality \( r(E) + 1 \) containing \( B \).

**Theorem.**

\[ \omega[B, r(E) + 1] = \sum_{k=1}^{r(E)} (-1)^{r(E)-k} \binom{r(E)}{k} c(k). \]

**Proof.** It follows from (2) and remark since

\[ \sum_{e \in E} \alpha(e, B) = \sum_{e \in E} \sum_{(F, e) \in A(B, e)} c[F, e] = \sum_{(F, e) \in A(B)} c[F, e]. \]

REFERENCES
