A HEREDITARY PROPERTY OF HM-SPACES

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Abstract. We have shown that a class of HM-spaces is invariant under projective topology, in particular, arbitrary product, subspace and separated quotient. We have also shown that \((E, \sigma(E, E'))\) is an HM-space for every locally convex space \((E, t)\) (see [2, Theorem 5.13]).

The nonstandard theory of topological vector spaces and in particular the construction of the nonstandard hull of an arbitrary topological vector space has been studied by Henson and Moore in [2] and [3]. We recount the principal ideas and definitions.

Let \((E, t)\) be a locally convex space and let \(^*\mathcal{M}\) be a nonstandard extension of a superstructure \(\mathcal{M}\) which contains \((E, t)\). An element \(p \in^* E\) is called \(t\)-finite if for every \(t\)-neighborhood \(U\) of zero there exists an integer \(n\) such that \(x \in n^*U\) and the set of \(t\)-finite elements of \(^*E\) is denoted by \(\text{fin}_t(^*E)\). The monad of \(O\) is defined by \(\mu_t(O) = \bigcap_{U \in \mathcal{U}}^*U\), where \(\mathcal{U}\) is a base of balanced, convex neighborhood of zero.

Both \(\text{fin}_t(^*E)\) and \(\mu_t(O)\) are vector spaces over the same field as \(E\). We denote the quotient vector space \(\text{fin}_t(^*E)/\mathcal{M}_t(O)\) by \(\hat{E}\), the canonical quotient mapping of \(\text{fin}_t(^*E)\) onto \(\hat{E}\) by \(\pi\) and the quotient topology defined on \(\hat{E}\) by \(\hat{t}\). The nonstandard hull of \((E, t)\) with respect to \(^*\mathcal{M}\) is the separated quotient space \((\hat{E}, \hat{t})\). Clearly, the map taking \(x\) to \(\pi(^*x)\) is a topological vector space isomorphism of \((E, t)\) into \((\hat{E}, \hat{t})\).

An element \(p \in^* E\) is called \(t\)-pre-near-standard if for each \(t\)-neighborhood \(U\) there exists \(y \in E\) such that \(x \in^* y + ^* U\) and the set of \(t\)-pre-near-standard elements of \(^*E\) is denoted by \(\text{pns}_t(^*E)\).

The nonstandard hull \((\hat{E}, \hat{t})\) of a locally convex space \((E, t)\) always contains \(\pi(\text{pns}_t(^*E))\) = the completion of \((E, t)\) and the nonstandard hulls determined by different nonstandard extensions \(^*\mathcal{M}\) need not even be isomorphic to each other. When every nonstandard hull \((\hat{E}, \hat{t})\) is equal to \(\pi(\text{pns}_t(^*E))\), we say that the nonstandard hulls of \((E, t)\) are invariant, i.e. we say that a locally convex space \((E, t)\) is an HM-space (see [4]).

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Throughout this paper \((E, t)\) will denote a separated locally convex space over \(K\), where \(K\) denotes the real or complex numbers.

Henson and Moore have shown the following [3]:

**Theorem 1** (Henson and Moore). The following conditions on a locally convex space \((E, t)\) are equivalent:

(a) The nonstandard hulls of \((E, t)\) are invariant;
(b) The nonstandard hull of \((E, t)\) is isomorphic to the completion of \((E, t)\) for every choice of the enlargement \(*\mathcal{M}\);
(c) \(\text{fin}_t(*E) = \text{psn}_t(*E)\) for every choice of \(*\mathcal{M}\);
(d) The nonstandard hull of \((E, t)\) is isomorphic to the completion of \((E, t)\) for some choice of the enlargement \(*\mathcal{M}\);
(e) \(\text{fin}_t(*E) = \text{psn}_t(*E)\) for some choice of \(*\mathcal{M}\);
(f) If \(\mathcal{F}\) is an ultrafilter on \(E\) and for each \(t\)-neighborhood \(U\) of zero there is an integer \(n\) such that \(nU \in \mathcal{F}\), then \(\mathcal{F}\) is a Cauchy filter.

For proofs of our theorems ([2], [3], [4] and [5]) we use a conditions (e) or (f) of preceding theorem.

**Theorem 2.** Every separated quotient \((E/H, t)\) of an HM-space \((E, t)\) is an HM-space.

**Proof.** Let \(*\mathcal{M}\) be a nonstandard extension of a superstructure \(\mathcal{M}\) which contains a space \((E, t)\). According to the condition (e) of the preceding theorem and [2, Theorem 1.2] it is sufficient to prove that \(\text{fin}_t(*E/H) \subset \text{psn}_t(*E/H)\). Let \(x \in \text{fin}_t(*E/H)\). Then, by [1, Definition 1.1] for each \(t\)-neighborhood \(U\) of zero there exists an integer \(n\) such that \(x \in n^*(U + *H) = n^*U + *H\), i.e. \(x \in \text{fin}_t(*E) + *H\). By assumption \((E, t)\) is an HM-space and then \(x \in \text{psn}_t(*E) + *H\). From this and [2, Theorem 1.2. iv] it follows that for every \(t\)-neighborhood \(U\) of zero there is \(y \in E\) such that \(x \in *y + *U + *H = *(y + H) + *(U + H)\), i.e. \(x \in \text{psn}_t(*E/H)\). Hence, the separated quotient space \((E/H, t)\) of a HM-space \((E, t)\) is an HM-space.

**Theorem 3.** A locally convex space \((E, t)\) is an HM-space, if and only if all its subspaces are HM-spaces.

**Proof.** The sufficiency follows immediately from [1, Theorem 4.6]. To prove the necessity we utilize condition (f) of Theorem 1. Let \(\mathcal{F}\) be an ultrafilter on its subspace \((H, t_H)\) with the property that for each \(t_H\)-neighborhood \(U\) of zero, there exists an integer \(n\) such that \(nU \in \mathcal{F}\) (\(t_H\) is the relative topology in \(H\)). By [1, chapter I (6), proposition 10] \(\mathcal{F}\) is the base of an ultrafilter on the space \((E, t)\). Suppose that \(NV \in \mathcal{F}\) for some \(N \in \mathcal{F}\). Then \(N(E \cap H) \in \mathcal{F}\) for some \(N \in \mathcal{F}\). By assumption \((E, t)\) is an HM-space and then \(\mathcal{F}\) is a base of Cauchy filter according to Theorem 1 (f). We want to show that \(\mathcal{F}\) is a Cauchy filter. If \(V\)
is a $t_H$-neighborhood of zero, there exists a $t$-neighborhood $U$ of zero such that $V \supset U \cap H$. For $t$-neighborhood $U$ of zero there exists $A \in \mathcal{F}$ such that $A - A \subset U$, i.e. $A \cap H - A \cap H \subset A - A \subset U$ and then $A \cap H - A \cap H \subset U \cap H \subset V$. Hence, for each $t_H$-neighborhood $V$ of zero there exists $B \in \mathcal{F}$ ($B = A \cap H$) such that $B - B \subset V$, i.e. $\mathcal{F}$ is a Cauchy filter on the space $(H, t_H)$, so the condition is necessary.

**Theorem 4.** The topological product of a family $(E_i, t_i)$ $i \in I$ of HM-spaces is an HM-space if and only if every space $(E_i, t_i)$ is an HM-space.

**Proof.** The necessity follows from the preceding theorem. To prove the sufficiency we use condition (e) of Theorem 1. Clearly, it is sufficient to prove that $\text{fin}_{t_i}^{*}(\prod_{i \in I} E_i) \subset \text{pns}_{t_i}^{*}(\prod_{i \in I} E_i)$, where $t$ is a product topology. Let $x \in \text{fin}_{t_i}^{*}(\prod_{i \in I} E_i)$ and let $U = \prod_{i=1}^{n} V_i \times \prod_{i \notin \{1, 2, \ldots, n\}} E_i$ be a neighborhood of zero of the space $\left(\prod_{i \in I} E_i, t_i\right)$. By [2, Definition 1.1] there exists an integer $n$ such that $x \in n^* U = n^* \left(\prod_{i=1}^{n} V_i \times \prod_{i \notin \{1, 2, \ldots, n\}} E_i \right)$. According to transfer principle it follows that $x(i) \in n^* V_i$ for $i \in \{1, 2, \ldots, n\}$ and $x(i) \in *E_i$ for $i \notin \{1, 2, \ldots, n\}$, i.e. for neighborhoods $V_i$ there exists $y(i) \in E_i$ such that $x(i) \in *y(i) + *V_i$ and $y(i) = O \in E_i$ for $i \notin \{1, 2, \ldots, n\}$. Hence, for each $t$-neighborhood $U$ of zero of the space $\left(\prod_{i \in I} E_i, t\right)$ there exists $y \in \prod_{i \in I} E_i$ such that $x \in *y + *U$, i.e. $x \in \text{pns}_{t_i}^{*}\left(\prod_{i \in I} E_i\right)$.

**Theorem 5.** Let $(E_i, t_i)$, $i \in I$, be a family of HM-spaces. Then the linear space $E$ equipped with a locally convex separated topology $t$ is an HM-space, if $t$ is the projective topology for $t_i$, $i \in I$.

**Proof.** Let $\mathcal{F}$ be an ultrafilter on $E$ with the property that for each $t$-neighborhood $V$ of zero, there exists an integer $n$ such that $nV \in \mathcal{F}$. By [1, Chapter I(6), Proposition 10] every $f_i(\mathcal{F})$ is a base ultrafilter for which the condition (f) of Theorem 1 holds. But, locally convex spaces $(E_i, t_i)$ are HM-spaces and therefore for every $t_i$-neighborhood $V_i$ of zero there exists $A \in \mathcal{F}$ such that $A - A \subset f_i^{-1}(f_i(A - A)) \subset f_i^{-1}(V_i)$, so $A - A \subset \bigcap_{i=1}^{n} f_i^{-1}(V_i)$. Hence, $\mathcal{F}$ is a Cauchy filter on the space $(E, t)$ and according to Theorem 1 (f), the theorem is proved.

**Corollary 1.** The projective limit of any family of HM-spaces is an HM-space.

**Proof.** The proof follows Theorem 5 and [5, chapter II.5].
COROLLARY 2. If \((E, t_i), i \in I\) is any family of a HM-spaces, then \((E, \sup t_i)_{i \in I}\) is an HM-space.

Proof. A topology \(\sup t_i, i \in I\) is projective for the systems \((E, t_i)\) and then the proof follows by the preceding theorem.

COROLLARY 3. For every locally convex space \((E, t)\), the associated space \((E, \sigma(E, E'))\) is an HM-space.

Proof. The weak topology is projective for the systems \((K_i, t_i)_{i \in E'}\), where \(K\) is the field of the real or complex numbers and \(t_i\) is the usual euclidean topology for every \(i \in E'\) (\(E'\) is the vector space of continuous linear functionals on a locally convex space \((E, t)\)). Hence, by the preceding theorem the space \((E, \sigma(E, E'))\) is an HM-space.

COROLLARY 4 [5, Chapter IV, (3.3)]. The space \((E, \tau_m)\) is an HM-space, if \(\tau_m\), is the minimal locally convex topology on \(E\).

Proof. According to [5] \(\tau_m = \sigma((E')^#, E')\) and by Corollary 3 \((E, \tau_m)\) is an HM-space.

Remark 1. By [3, Theorem 2] and [5, Chapter II. 6, Example 1] it follows that the class of HM-spaces is not invariant under direct sums and inductive limit.

Remark 2. According to Corollary 3, it is easy to see that the barrelled (quasi-barrelled, bornological, ultra-bornological) space associated to an HM-space is not an HM-space, in general. (About associated spaces see [6]).

REFERENCES


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