ON TWO OPEN PROBLEMS OF CONTRACTIVE MAPPINGS

V. Totik

Abstract. Two open problems are solved concerning the fixed points of contractive mappings. The first is an example of a shrinking mapping of the closed unit ball in a Banach space without any fixed point. The second solves a question of B. Fischer.

1. Let \( (X, d) \) be a metric space, \( T : X \to X \) a mapping of \( X \) into itself. \( T \) is said to be shrinking if \( d(Tx, Ty) < d(x, y) \) for every \( x, y \in X \).

It is well known (see e.g. [3]) that if \( X \) is compact and \( T : X \to X \) is a shrinking mapping, then \( T \) has a fixed point. By a beautiful theorem of Browder [1] the same conclusion holds provided \( X \) is the closed unit ball of a Hilbert space and \( T \) is shrinking. In connection with these results D. R. Smart raised the following question [3, p. 39]: “Does every shrinking mapping of the closed unit ball in a Banach space have a fixed point?” The aim of this paragraph is to give a negative answer to this problem.

Theorem 1. There exists a Banach space \( B \) and an affine shrinking mapping \( T \) of the closed unit ball \( U \) of \( B \) into the boundary \( \partial U \) of \( U \) such that \( T \) does not have any fixed point.

Proof. Let \( c_0 = \{ x = \{ x_i \}^\infty_1 \mid \lim_{i \to \infty} x_i = 0 \} \) be the space of real sequences converging to 0 with norm \( \| x \| = \sup_i |x_i| \). Let \( B = c_0 \) and \( T(x_1, x_2, \ldots, x_n, \ldots) = (1, x_2/2 + 1/2, 2x_3/3 + 1/3, \ldots, (1-1/n)x_n + 1/n, \ldots) \) i.e. \( T \) is defined by \( (Tx)_n = (1-1/n)x_n + 1/n \). If \( U \) is the unit ball in \( B \), then clearly \( T : U \to \partial U \) and \( T \) is affine. \( T \) is shrinking. Let \( x = \{ x_i \}^\infty_1, \ y = \{ y_i \}^\infty_1, \ x \neq y \). Then \( 0 < \varepsilon := \| x - y \| = | x_{n_0} - y_{n_0} | \) for some \( n_0 \). Let \( N > 2 \) be so large that the inequalities

\[ |x_n| < \varepsilon/4, \ |y_n| < \varepsilon/4 \quad (n \geq N) \]

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be satisfied. Now
\[
|(T_x)_i - (T_y)_i| = (1 - 1/i)|x_i - y_i| \leq \begin{cases} 
2\varepsilon/4 = \varepsilon/2 & \text{if } i \geq N \\
(1 - 1/N)|x_i - y_i| \leq |1 - 1/N|\varepsilon & \text{if } i < N
\end{cases}
\]
i.e.
\[
||T x - T y|| \leq (1 - 1/N)\varepsilon,
\]
and so \(T\) is really a shrinking mapping.

Finally \(T\) does not have any fixed point: \(x = \{x_i\}_1^\infty\) where a fixed point of \(T\), then we would have
\[
(1 - 1/i)x_i + 1/i = (T x)_i = x_i
\]
i.e. \(x_i = 1\) for all \(i\), but the sequence \(\{1\}_1^\infty\) does not belong to \(B = c_0\).

We have proved our theorem.

2. In [2] B. Fischer made the following conjecture. Suppose \(S\) and \(T\) are mapping of the complete matrix space \(X\) into itself, with either \(S\) or \(T\) continuous, satisfying the inequality
\[
(1) \quad d(Sx, TSy) \leq c \text{diam} \left\{x, Sx, Sy, TSy\right\}
\]
for all \(x, y\) in \(X\), where \(0 \leq c < 1\). Then \(S\) and \(T\) have a unique common fixed point.

This conjecture has been open even for compact \(X\). Now we show that it is true for \(c < 1/2\) but false for \(c \geq 1/2\).

**Theorem 2.** If \(X\) is complete, \(S: X \to X, T : X \to X\) with property (1), where \(c < 1/2\), then \(S\) and \(T\) have a unique common fixed point. On the other hand, there are a four point \(X\) and \(S: X \to X, T : X \to X\) mappings of \(X\) without fixed point satisfying
\[
d(Sx, TSy) \geq 1/2 \ \text{diam} \ \{x, Sx, Sy, TSy\}.
\]

Thus, if \(\alpha < 1/2\) we do not need any continuity assumption, and for \(\alpha \geq 1/2\) even the simultaneous continuity of \(S\) and \(T\) and the compactness of \(X\) do not help.

**Proof.** To prove the first part of our theorem let \(x_0 \in X\) be arbitrary and let
\[
x_n = \begin{cases} 
(TS)^{n/2}x_0, & \text{if } n \text{ is even} \\
S(TS)^{(n-1)/2}x_0, & \text{if } n \text{ is odd}.
\end{cases}
\]
By (1)
\[
d(x_{2n+1}, x_{2n}) = d(STSx_{2n-2}, TSx_{2n-2}) \leq c \text{diam} \{Sx_{2n-2}, TSx_{2n-2}, STSx_{2n-2}\} = \\
= c \text{diam} \{x_{2n-1}, x_{2n}, x_{2n+1}\} \leq c(d(x_{2n}, x_{2n-1}) + d(x_{2n+1}, x_{2n}))
\]
and thus

(2) \[ d(x_{2n+1}, x_{2n}) \leq (c/(1 - c))d(x_{2n}, x_{2n-1}) \quad (n \geq 1) \]

Similarly,

\[
d(x_{2n+2}, x_{2n+1}) = d(Sx_{2n}, TSx_{2n}) \leq c \operatorname{diam} \{x_{2n}, x_{2n+1}, x_{2n+2}\} \leq c(d(x_{2n+1}, x_{2n}) + d(x_{2n+2}, x_{2n+1}))
\]

by which

(3) \[ d(x_{2n+2}, x_{2n+1}) \leq (c/(1 - c))d(x_{2n+1}, x_{2n}) \]

Since \( c < 1/2 \) we have \( c/(1 - c) < 1 \), and so (2) and (3) imply that the sequence \( x_n \) is a Cauchy sequence and thus, by completeness, \( x_n \to z (n \to \infty \in X) \). Using again (1) we get

\[
d(Sz, x_{2n+2}) \leq c \operatorname{diam} \{z, Sz, x_{2n+1}, x_{2n+2}\} \leq c(d(Sz, z) + d(z, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}))
\]

Letting here \( n \to \infty \) we obtain \( d(Sz, z) \leq cd(Sz, z) \) i.e. \( d(Sz, z) = 0 \), \( Sz = z \). But then

\[
d(z, Tz) = d(Sz, TSz) \leq c \operatorname{diam} \{z, Sz, TSz\} = c(d(z, Tz)
\]

i.e. \( d(z, Tz) = 0 \), \( Tz = z \) and thus \( z \) is a common fixed point of \( S \) and \( T \). The uniqueness of the common fixed point follows easily from (1).

After this let us prove that the conjecture is false for \( c = 1/2 \) and hence also \( c \geq 1.2 \). Let \( X = \{A, B, C, D\} \) with \( d(A, D) = d(B, C) = d(B, D) = 1 \) and \( d(A, B) = d(C, D) = 2 \) (see the first figure) and let \( S \) and \( T \) be the two mapping indicated below:

Neither \( S \) nor \( T \) have any fixed point. However, \( Sx \in \{D, C\}, TSy \in \{A, B\} \) and so \( d(Sx, TSy) = 1 \) for every \( x, y \in X \); furthermore

a) \[ d(x, Sx) = 2, \quad \text{if } x = C \text{ or } x = D \]
b) \[ d(Sx, Sy) = 2, \quad \text{if } x + A \text{ and } y \in \{B, D\} \ or \ x = B \text{ and } y \in \{A, C\} \]
c) \[ d(x, TSy) = 2, \quad \text{if } x = A \text{ and } y \in \{A, C\} \text{or } x = B \text{ and } y \in \{B, D\} \]
i.e. in any case $\text{diam}\{x, Sx, Sy, TSy\} = 2$ and so (1) holds for every $x, y \in X$ with $c = 1/2$.

We have proved our theorem.

REFERENCES


Százg, Bolyai Institute
Aradi vétanuk tere 1
Hungary

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