ON THE AUTOMORPHISM GROUP OF AN INFINITE GRAPH

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Abstract. In this paper, a specially defined automorphism group \( \Gamma(G) \) of a connected countable simple infinite graph is considered. As the main result, we prove that \( \Gamma(G) \) contains at most one non-trivial element. All infinite graphs with a non-trivial automorphism group are completely described.

Finally, for graphs with odd, or with a small even number (2 or 4) of non-zero eigenvalues, the corresponding automorphism groups are characterized.

1. Introduction. Throughout the paper, \( G \) is a connected infinite countable graph without loops or multiple edges, which we briefly call a graph. Its vertex set is \( V(G) = N \), and its adjacency matrix \( A = [a_{ij}] \) is an infinite \( N \times N \) matrix, where

\[
a_{ij} = \begin{cases} 
a^{i+j-2} & \text{if } i, j, \text{ are adjacent} \\
0 & \text{otherwise}
\end{cases}
\]

(\( a \) is a fixed positive constant, \( 0 < a < 1 \)).

Hence, the whole graph \( G \) is labelled and the “weight” of vertex \( v_i = i \) is \( a^{i-1} (i \in N) \).

For other definitions and results concerning spectra of infinite graphs, one can see [3, 4, 5].

2. Results. The automorphism group \( \Gamma(G) \) of an infinite graph \( G \) defined here, depends on the matrix \( A \), thus especially depends on the way of labelling of the vertex set \( V(G) \).

Namely, we put \( P \in \Gamma(G) \) of and only if

\[
(1) \quad AP = PA,
\]

where \( P = [p_{ij}] \) is an infinite permutation matrix of the set \( V(G) \),

\[
p_{ij} = \begin{cases} 
1, & j = \omega(i) \\
0, & j \neq \omega(i)
\end{cases}
\]

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and $\omega$ is the corresponding permutation of the set $N$.

In the sequel, we identify any automorphism $P \in \Gamma(G)$ with the corresponding permutation of the set $N$.

Obviously, each permutation $P \in \Gamma(G)$ is a unitary operator in the corresponding Hilbert space $H$, and

$$P e_i = e_{\omega(i)} \quad (i \in N),$$

for a fixed orthonormal basis $\{e_i\}_i^\infty$ of $H$.

Relation (1) is equivalent to

$$a_{\omega(i)\omega(j)} = a_{ij} \quad (i, j \in N),$$

so that vertices $i, j$ are adjacent if and only if $\omega(i), \omega(j)$ are adjacent. In this case (2) gives

$$a^{\omega(i)+\omega(j)-2} = a^{i+j-2}, \text{ or}$$

$$\omega(i) - i = -[\omega(j) - j] \quad (i, j, \text{ adjacent}).$$

The last relation is very restrictive, and it is the main difference in comparison to the finite case.

**Lemma 1.** (i) For every $\omega \in \Gamma(G)$ there is a unique integer $d = d(\omega)$ such that

$$| \omega(i) - i | = d \quad (i \in N).$$

(ii) If $\omega(i) = i$ for an $i \in N$, then $\omega = \text{id}$.

(iii) If $G$ has at least one odd cycle, then $\Gamma(G)$ is trivial.

**Proof.** (i) For any two adjacent vertices $i, j \in V(G)$, relation (3) yields

$$| \omega(i) - i | = | \omega(j) - j |,$$

and the connectivity of $G$ ends the proof.

The last two statements are then immediate by (i). □

**Examples.** (1) The automorphism group of the one-way infinite path

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  i_1  i_2  i_3  i_4  ...  
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is always trivial.

Indeed, for any $\omega \in \Gamma(G)$ we must have $\omega(i_1) = i_1$, thus $\omega = \text{id}$.

(2) The corresponding group of an infinite two-way infinite path

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  ...  j_2  j_1  i_1  i_2  ...  
```
is either trivial or contains exactly one non-trivial element. If, for example, $i_p = 2p - 1$ and $j_p = 2p(p \in N)$, then it is non-trivial.

The following property is one of the most important properties of the groups considered.

**Theorem 1.** In each case $|\Gamma(G)| \leq 2$.

**Proof.** Let $\omega \in \Gamma(G)$ and $d = \omega(1) - 1$. Then $|\omega(i) - i| = d$ $(i \in N)$, and the only possibilities we have are

$$
\omega(1) = d + 1, \quad 1 = \omega(d + 1), \ldots, \quad \omega(d) = 2d, \quad d = \omega(2d), \quad \omega(2d + 1) = 3d + 1, \\
2d + 1 = \omega(3d + 1), \ldots
$$

Generally, we obtain

$$
\omega(i) = i + (-1)^{(i-1)/d} d.
$$

If now $\omega_1$, $\omega_2 \in \Gamma(G)$ are the automorphisms with the corresponding values $d_1 = d_2$, by the last relation we immediately find $\omega_1 = \omega_2$.

Next, let $\omega_1$, $\omega_2$ be the different automorphisms with $d_1 < d_2$. Then we get

$$
\omega_2 \omega_1(1) = \omega_2(1 + d_1) = 1 + d_1 + d_2,
$$

and also

$$
\omega_2 \omega_1(d_1 + 1) = \omega_2(1) = 1 + d_2,
$$

whence $d_1 = d_2 = d_2 - d_1$; thus $d_1 = 0$, $\omega_1 = \text{id}$, q.e.d.

Hence, $\Gamma(G)$ is always either trivial or contains at most one non-trivial element (which is then—involution). $\square$

So, the most important question concerning $\Gamma(G)$ is when it is non-trivial. In the next main theorem we completely describe all the infinite graphs which have a non-trivial automorphism group. It appears that the considered property depends only on the structure of the graph, and on the way of labelling of its vertex set.

First, let $G$ be any bipartite graph. Its characteristic parts are denoted by $N_1$ and $N_2$, assuming always that the minimal element is in $N_1$. Note that $N_1, N_2$ are not the cardinals, but the corresponding sets of indices.

Next, we need the notion of symmetric bipartite graphs (briefly, SBGs). We call an infinite bipartite graph with the characteristic parts $N_1, N_2$—symmetric, if there is a bijection $\pi : N_1 \to N_2$ such that two vertices $a \in N_1, \pi(b) \in N_2$ are adjacent if and only if the vertices $b \in N_1, \pi(a) \in N_2$ are.

If $G$ is a SBG, then obviously $N_1, N_2$ are infinite.

If, additionally, we have that $\pi(a) - a = d = \pi(1) - 1$ for each $a \in N_1$, we say that $N_1, N_2$ are good. In this case, we obtain that

$$
N_1 = \{(2s - 2)d + r \mid r \leq d, \; s \in N\}, \quad N_2 = \{(2s - 1)d + r \mid r \leq d, \; s \in N\}.
$$
Theorem 2. Graph $G$ has a non-trivial automorphism group if and only if it is a SBG with the good characteristic parts $N_1$ and $N_2$.

Proof. Let $\Gamma(G)$ be non-trivial, and let $\omega \in \Gamma(G)$ be the unique non-trivial automorphism (involution) of $G$. Then, by Lemma 1(iii), $G$ can not have any odd cycle as an induced subgraph, thus it must be bipartite.

Let, next, the characteristic parts of $G$ be $N_1, N_2$ with the minimal element in $N_1$. Then, by the odd-path and the even-path characterizations of $N_1, N_2$, we easily find that

$$\omega(a) = a + d(a \in N_1), \quad \omega(b) = b - d \quad (b \in N_2),$$

where $d = d(\omega) = \omega(1) - 1 > 0$.

Hence, $N_1, N_2$ are good, and $\omega$ is a needed bijection between $N_1$ and $N_2$.

Since the converse statement is immediate, this completes the proof. ∎

As examples, we consider the infinite graphs with a finite number $p(p \geq 2)$ of non-zero eigenvalues.

Proposition 1. Let $G$ have an odd number of non-zero eigenvalues. Then its automorphism group is always trivial.

Proof. As is known ([4]), each bipartite infinite graph, for every $a \in (0, 1)$, has the spectrum symmetric about the zero. Hence, if $G$ has an odd number of non-zero eigenvalues, it cannot be bipartite, whence $\Gamma(G)$ is trivial. ∎

Next, consider the infinite graphs with $p = 2$ or 4 non-zero eigenvalues.

We need the notion of characteristic subsets of $G$. The characteristic subsets $N_1, N_2, \ldots$ of an infinite graph are the equivalence classes related to the equivalence relation on the vertex set $N : x \sim y$ if and only if $x, y$ are not adjacent and they have the same neighbors. Their number is finite or infinite and always greater than 1. If it is finite, $G$ is said to be of finite type (type $p$, if this number is $p$) [5]. The corresponding quotient graph is denoted by $g$, and often called the canonical graph of $G$. If, for example, $G$ is the complete $m$-partite graph $K(N_1, \ldots, N_m)(m \geq 2)$, then its characteristic subsets will be $N_1, \ldots, N_m$, and its canonical graph is $K_m$.

Lemma 2. (i) If $\omega(x) \in N_i$ for an $x \in N_i$, then $\omega = \text{id}$.

(ii) If $G$ is of finite type $p$ and $\Gamma(G)$ is non-trivial, then $p$ is even.

Proof. (i) Assume, on the contrary, $\omega \neq \text{id}$, and denote by $M_1, M_2$ the characteristic parts of $G$. Since if follows easily that each $N_i$ is contained either in $M_1$ or in $M_2$, we get the statement.

(ii) Let $\omega$ be the non-trivial automorphism of $\Gamma(G)$. Since by (i), $\omega$ is an involution on the set $\{N_1, \ldots, N_p\}$, without fixed elements, we have that $p$ must be even. ∎
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PROPOSITION 2. Let $G$ have exactly two non-zero eigenvalues. Then $\Gamma(G)$ is non-trivial iff $G$ is a complete bipartite graph with good characteristic parts.

Proof. In [4], we proved that $G$ has exactly two non-zero eigenvalues if and only if it is a complete bipartite graph. Hence, $\Gamma(G)$ is non-trivial if $N_1, N_2$ are good (and consequently-infinite). □

PROPOSITION 3. The following graph

$$
\begin{array}{c}
\circ \quad \circ \quad \circ \quad \circ \\
N_1 \quad N_2 \quad N_3 \quad N_4
\end{array}
$$

where $N_4 = N_1 + d$, $N_3 = N_2 - d$ ($d \neq 0$) is the unique connected infinite graph with four non-zero eigenvalues and a non-trivial automorphism group.

Proof. In [5] we proved that $G$ has exactly four non-zero eigenvalues if and only if its canonical graph is one of the eight particular graphs with 4, 5 or 6 vertices. Since six of them have a triangle as a subgraph, their automorphism groups must be trivial. Since next, the seventh of them is $P_5$ with 5 characteristic subsets, by Lemma 2 (ii), its automorphism group is trivial, too. Hence, only $P_4$ remains, and the remaining proof is easy. □

The general problem for any even number $p$ of non-zero eigenvalues ($p \geq 6$) is obviously equivalent to the determination of all finite connected canonical symmetric bipartite graphs with exactly $p$ non-zero eigenvalues. The present author thinks it can be solved at least for $p = 6$, and may be for $p = 8$.

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REFERENCES


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