ON GRAPHS WHOSE SPECTRAL SPREAD DOES NOT EXCEED 4

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Abstract. In this paper all minimal graphs with the property of having the spectral spread greater than 4 are determined. In addition, all connected graphs whose spectral does not exceed 4 are described.

1. Introduction

All considered graphs are undirected graphs without loops or multiple edges. By eigenvalues of a graph $G$ we mean eigenvalues of its 0–1 adjacency matrix $A$. The spectral spread (briefly the spread) $s(G)$ of $G$ is the spread $s(A)$ of its adjacency matrix $A$, i.e. $s(G) = s(A) = r(G) - \lambda(G)$, where $r(G)$ and $\lambda(G)$ are the largest and the least eigenvalue of $G$. For definition and properties of the spread of matrices, one can consult [3].

J. H. Smith [4] has determined all graphs whose largest eigenvalue does not exceed 2; after that D. Cvetković, M. Doob and I. Gutman [1] have determined all minimal graphs with the property of having the largest eigenvalue greater than 2. In this paper we extend their results in some sense.

Let $H$ be a proper induced subgraph of $G$. By virtue of the well known Interlacing Theorem (see [2] for example) it follows that $r(G) \geq r(H)$ and $\lambda(G) \leq \lambda(H)$, i.e. $s(G) \geq s(H)$. Thus for any real number $L > 0$ we may consider the graphs with $s(G) > L$ that are minimal with respect to that property. In this paper we shall find all such graphs for $L = 4$.

We consider also the following question: For a real number $L > 0$ find all graphs with $s(G) \leq L$. We give an explicit description of the connected graphs $G$ satisfying $s(G) \leq 4$. Combining this with the results of Smith [4], we see that there are exactly five graphs with $r(G) > 2$ and $s(G) \leq 4$.

2. Minimal connected graphs with $s(G) > 4$

Recall that a graph is minimal with respect to (briefly w.r.t.) the property $P$ if it has the property $P$ and none of its proper induced subgraphs has this property.

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Now we determine all connected minimal graphs w.r.t. the property of having the spread greater than 4.

The determination of minimal trees with the spread greater than 4 is equivalent to the determination of minimal trees with the largest eigenvalue greater than 2.

**Lemma 1.** (Cvetković, Doob, Gutman [1]) *There are exactly 9 minimal trees w.r.t. the property of having the spread greater than 4 and these are the graphs $G_{22} - G_{30}$ displayed in Fig. 1. ☐*

**Theorem 1.** *There are exactly 30 minimal connected graphs w.r.t. the property of having the spread greater than 4 and they are displayed in Fig. 1.*

![Graphs](image)

**Proof.** It is easy to check that the graphs $G_1 - G_{30}$ in Fig. 1 are minimal graphs w.r.t. the property of having the spread greater than 4.

We prove that if a connected graph $G$ is a minimal graph with the spread greater than 4, then $G$ is one of the graphs $G_1 - G_{30}$ in Fig. 1. Lemma 1 takes
care of the case when $G$ is a tree. Hence, it is enough to consider the graphs with circuits.

Let $G$ be a connected minimal graph with the spread greater than 4 and let $n$ be the length of the shortest circuits of $G$. We denote the vertices of an arbitrary circuit $C_n$ of $G$ by $v_1, \ldots, v_n$, so that the vertices $v_i$ and $v_{i+1}$ ($i = 1, \ldots, n-1$), $v_1$ and $v_n$ are adjacent. Let $T_{i_1 \ldots i_k}$ ($1 \leq i_1 < \cdots < i_k \leq n$; $1 \leq k \leq n$) be the set of vertices from $V(G) \backslash V(C_n)$ which are adjacent exactly to the vertices $v_{i_1}, \ldots, v_{i_k}$ of $C_n$. Let $T_0$ be the set of vertices from $V(G) \backslash V(C_n)$ which are not adjacent to any vertex of $C_n$. We distinguish the following five cases:

**Case 1:** $n = 3$. If at least one of the sets $T_{ij}$ ($1 \leq i < j \leq 3$) is nonempty, then the graph $G_1$ is an induced subgraph, and hence equals $G$. Let $T_{ij} = \emptyset$ ($1 \leq i < j \leq 3$). Then if $|T_{123}| \geq 2$ and no two vertices of $T_{123}$ are adjacent, $G$ contains the proper induced subgraph $G_1$ contradicting the minimality condition. Thus, if $|T_{123}| \geq 2$, the graph $G_2$ is contained in (and hence equals) $G$. If $|T_{123}| = 1$, then $G$ is $G_3$. Let $T_{123} = \emptyset$. If some of the sets $T_i$ ($1 \leq i \leq 3$) contains more than one vertex, then $G$ is either $G_4$ or $G_5$. Let $|T_i| \leq 1$ ($1 \leq i \leq 3$). Having in mind symmetric cases, we distinguish the following cases:

1$^	ext{o}$ | $T_1 = 1$, $T_2 = T_3 = \emptyset$. If the set $T_0$ contains two vertices adjacent to the vertex $x \in T_1$, then $G$ is either $G_6$ or $G_7$. If $T_0$ contains only one vertex adjacent to the vertex $x \in T_1$, then $G$ is $G_8$.

2$^	ext{o}$ | $T_1 = |T_2| = 1$, $T_3 = \emptyset$. If the vertices $x \in T_1$ and $y \in T_2$ are adjacent, then $G$ is $G_9$. If the vertices $x$ and $y$ are not adjacent, then $G$ is either $G_{10}$ or $G_{11}$.

3$^	ext{o}$ | $T_1 = |T_2| = |T_3| = 1$. If at least two vertices between $x \in T_1$, $y \in T_2$ and $z \in T_3$ are adjacent, then $G$ contains the proper induced subgraph $G_9$ contradicting the minimality condition. Therefore $G$ is $G_{12}$.

**Case 2:** $n = 4$. Then $T_{12} = T_{14} = T_{23} = T_{34} = T_{123} = T_{124} = T_{134} = T_{234} = T_{1234} = \emptyset$ because $G$ has not triangles. If at least one of the sets $T_i$ ($1 \leq i \leq 4$) is nonempty, then $G$ is $G_{13}$. Let $T_i = \emptyset$ ($1 \leq i \leq 4$). If $T_{12} \neq \emptyset$ and $T_{24} = \emptyset$, then $G$ is $G_{14}$. If $T_{13} \neq \emptyset$ and $T_{24} \neq \emptyset$, then $G$ contains the proper induced subgraph $G_{14}$ contradicting the minimality condition.

In the orders cases ($n \geq 5$) we have that $T_{i_1 \ldots i_k} = \emptyset$ ($1 \leq i_1 < \cdots < i_k \leq n$, $2 \leq k \leq n$) because $G$ does not contain circuits whose length is less than $n$. If at least one of the sets $T_i$ ($1 \leq i \leq n$) contains more than one vertex, then $G$ contains the proper induced subgraph $G_{23}$ contradicting the minimality condition. Let $|T_i| \leq 1$ ($1 \leq i \leq n$).

**Case 3:** $n = 5$. Now, if at least two of the sets $T_i$ ($1 \leq i \leq 5$) are nonempty, then $G$ is one of the graphs $G_{15}, G_{16}, G_{17}$. If exactly one of the sets $T_i$ ($1 \leq i \leq 5$) is nonempty, then $G$ is $G_{18}$.

**Case 4:** $6 \leq n \leq 8$. Then for $n = 6,7,8$ the graphs $G_{19}, G_{20}$ and $G_{21}$, respectively, are contained in (and hence are equal to) $G$. 


Case 3: \( n \geq 9 \). Then \( G \) contains the proper induced subgraph \( G_{20} \) contradicting the minimality condition.

This completes the proof of Theorem 1. \( \square \)

3. Graphs whose spread does not exceed 4

In this section we determine all connected graphs whose spread does not exceed 4. In the proof of Theorem 2 we use the following lemma.

**Lemma 2.** (Smith [4]) Let \( G \) be a graph with the largest eigenvalue \( r(G) \). Then \( r(G) \leq 2 \) if and only if each component of \( G \) is an induced subgraph of one of the graphs \( H_6 - H_{11} \) displayed in Fig. 2. \( \square \)

**Theorem 2.** Let \( G \) be a connected graph with the spread \( s(G) \). Then \( s(G) \leq 4 \) if and only if \( G \) is an induced subgraph of one of the graphs displayed in Fig. 2.

![Fig. 2]

**Proof.** Lemma 2 takes care of the case when \( G \) is a tree. Namely the determination of the trees with the spread less than or equal to 4 is equivalent to the determination of the trees whose largest eigenvalue does not exceed 2. Hence, it is enough to consider the graphs with circuits.

It is easy to check that the graphs \( H_1 - H_{11} \) have the spread less than or equal to 4. Consequently, each induced subgraph of these graphs has the spread less than or equal 4.

Conversely, let \( G \) be a connected graph with circuits, whose spread does not exceed 4. To describe \( G \), we use the method of impossible subgraphs. We note that \( G \) does not contain any of the graphs \( G_1 - G_{30} \) depicted in Fig. 1, as an induced subgraph. The denotation and scheme of the proof are the same as in Theorem 1.

Let \( C_n \) be the smallest circuit in \( G \). We distinguish the following four cases:
Case 1: $n = 3$. We have $T_{ij} = \emptyset$ ($1 \leq i < j \leq 3$) since otherwise $G$ would contain the induced subgraph $G_1$. Moreover, the sets $T_i$ ($1 \leq i \leq 3$) can contain at the most one vertex (otherwise $G$ would contain $G_4$ or $G_5$ as an induced subgraph); thus, $|T_i| \leq 1$ ($1 \leq i \leq 3$). Next, the set $T_{123}$ can contain at the most one vertex, too (otherwise $G$ would contain $G_1$ or $G_2$ as an induced subgraph).

By a direct verification, we determine the mutual relations between the corresponding sets. First, the vertices of $T_i$ and $T_j$ ($1 \leq i < j \leq 3$) cannot be adjacent (otherwise $G$ would contain $G_9$ as an induced subgraph). Next, the sets $T_i$ and $T_{123}$ are not consistent (otherwise $G$ would contain $G_1$ or $G_3$ as an induced subgraph).

Taking into account all possible combinations and having in mind symmetry, we distinguish the following subcases:

1° $T_1 = T_2 = T_3 = T_{123} = \emptyset$. Then $G = G_3$.

2° $|T_1| = 1$, $T_2 = T_3 = T_{123} = \emptyset$. Then $|T_0| \leq 1$, because otherwise $G$ would contain at least one of the graphs $G_6$, $G_7$, $G_8$ as an induced subgraph. Thus, either $G = H_1$ or $G = H_2$.

3° $|T_{123}| = 1$, $T_1 = T_2 = T_3 = \emptyset$. In this case $T_0 = \emptyset$ (otherwise $G$ would contain $G_3$ as an induced subgraph). Thus, $G = H_4$.

4° $|T_1| = |T_2| = 1$, $T_3 = T_{123} = \emptyset$. Then $T_0 = \emptyset$, because otherwise $G$ would contain $G_{10}$ or $G_{11}$ as an induced subgraph. Hence, $G = H_3$.

We note that the combination $|T_1| = |T_2| = |T_3| = 1$, $T_{123} = \emptyset$ is impossible. Indeed, in the contrary case $G$ would contain $G_{12}$ as an induced subgraph.

Case 2: $n = 4$. Then $T_i = \emptyset$ ($1 \leq i \leq 4$), because otherwise $G$ would contain $G_{13}$ as an induced subgraph. Besides, $T_{13} = T_{24} = \emptyset$, because in the contrary case $G$ would contain $G_{14}$ as an induced subgraph. Thus, $G = G_4$.

Case 3: $n = 5$. Then $|T_i| \leq 1$ ($1 \leq i \leq 5$), because otherwise $G$ would contain $G_{23}$ as an induced subgraph. Moreover, the sets $T_i$ and $T_j$ ($1 \leq i < j \leq 5$) are not consistent (in the contrary case $G$ would contain at least one of the graphs $G_{15}$, $G_{16}$, $G_{17}$ as an induced subgraph). Thus, we have only two possibilities:

1° $T_i = \emptyset$ ($1 \leq i \leq 5$). Then $G = G_5$.

2° $|T_1| = 1$, $T_i = \emptyset$ ($2 \leq i \leq 5$). In this case $T_0 = \emptyset$, because otherwise $G$ would contain $G_{18}$ as an induced subgraph. Thus, $G = H_5$.

Case 4: $n \geq 6$. Then $T_i = \emptyset$ ($1 \leq i \leq n$). Indeed, in the contrary case $G$ would contain the graphs $G_{19}$, $G_{20}$ and $G_{21}$ for $n = 6, 7, 8$, respectively, and the graph $G_{20}$ for $n \geq 9$. Thus, $G = C_n$.

This completes the proof of the theorem. $\square$

Corollary. There are exactly five connected graphs with $r(G) > 2$ and $s(G) \leq 4$ and these are the graphs $H_1 - H_5$ displayed in Fig. 2.
4. Minimal disconnected graphs with \( s(G) > 4 \)

In this section we determine all disconnected minimal graphs w.r.t. the property of having the spread greater than 4.

Let \( C_n \) be a circuit of length \( n \), \( P_n \) a path of length \( n-1 \), \( S_n \) a star with \( n+1 \) vertices and \( V_{m,n} \) and \( W_n \) the graphs displayed in Fig. 3.

\[
\begin{align*}
\text{Fig. 3}
\end{align*}
\]

**Theorem 3.** There are exactly 17 disconnected minimal graphs w.r.t. the property of having the spread greater than 4. These are

\[
\begin{align*}
H_1 \cup C_n, & \quad H_1 \cup C_6, & \quad H_1 \cup P_7, & \quad H_1 \cup V_{1,2}, & \quad H_1 \cup S_4, \\
H_2 \cup P_8, & \quad H_3 \cup P_9, & \quad H_3 \cup S_3, & \quad H_4 \cup P_3, & \quad H_5 \cup C_4, \\
H_5 \cup C_6, & \quad H_5 \cup C_8, & \quad H_5 \cup P_9, & \quad H_5 \cup V_{1,3}, & \quad H_5 \cup V_{2,2}, \\
H_5 \cup W_2, & \quad H_5 \cup S_4.
\end{align*}
\]

**Proof.** It is easy to check that all graphs (1) are minimal graphs w.r.t. the property of having the spread greater than 4.

Let \( G \) be any disconnected minimal graphs with the spread greater than 4. Then \( G \) satisfies the following conditions:

1) \( G \) has exactly two component, i.e. \( G = G_1 \cup G_2 \) and \( r(G_1) \neq r(G_2) \), \( \lambda(G_1) \neq \lambda(G_2) \) hold. Supposing \( r(G_1) > r(G_2) \), we have that \( \lambda(G_1) > \lambda(G_2) \).

2) Each component has the property

\[
s(G_i) = r(G_i) - \lambda(G_i) \leq 4 \quad (i = 1, 2).
\]

By Theorem 2 we conclude that each component is an induced subgraph of one of the graphs displayed in Fig. 2.

3) At least one of components has the largest eigenvalue greater than 2.

4) \( G_1 \) is one of the graphs \( H_1 - H_9 \) from Fig. 2, because they are only connected graphs which satisfy 2) and 3).

5) \( G_2 \) is a minimal graph w.r.t. the property \( \lambda(G_2) < r(G_1) - 4 \).

We distinguish the following five cases:

1° \( G_1 = H_1 \). Then \( G_2 \) is one of the graphs \( C_4, C_6, P_7, V_{1,2} \) and \( S_4 \), because they are only graphs which satisfy conditions 2) and 5).

2° \( G_1 = H_2 \). Since \( H_1 \) is a proper induced subgraph of \( H_2 \), then \( G_2 \) satisfies 2), 5) and relation

\[
\lambda(G_2) > r(H_1) - 4.
\]
The graph $P_6$ is the unique graph satisfying all the above conditions.

$3^\circ \ G_1 = H_3$. In this case $H_1$ is a proper induced subgraph of $H_3$, too, so $G_2$
must satisfy the conditions 2), 5) and (2). The graphs $P_5$ and $S_3$ are the unique such graphs.

$4^\circ \ G_1 = H_4$. Then $P_3$ is the unique graph satisfying 2) and 5).

$5^\circ \ G_1 = H_5$. Then $G_2$ is one of the graphs $C_4, C_6, C_8, P_3, V_{1,3}, V_{2,2}, W_2$ and $S_4$, since they are the unique graphs satisfying 2) and 5).

This completes the proof Theorem 3. □

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