THE \((\psi, \xi, \eta, \mathfrak{g})\) STRUCTURE ON SUBSPACES OF THE SPACE WITH
THE \(\varphi(4, -2)\) STRUCTURE

Jovanka Nikič

Abstract. Let a tensor field \(\varphi\), \(\varphi \neq 0\), \(\varphi \neq 1\), of type \((1,1)\) and of class \(C^\infty\) be given on \(M^n\) such that \(\varphi^4 - \varphi^2 = 0\), and rank \(\varphi = n - 1\). The structure \(\Phi = 2\varphi - 1\) is an almost product structure. \(\Phi\) induces on hypersurface \(K\) a Sato structure. In this paper it is proved that the structure Sato \(\psi\) induced by \(\Phi\) on \(K^*\) is equal to the \(\overline{\varphi}\) (\(\overline{\varphi}\) is the restriction of the structure \(\varphi\) on \(K^*\)).

Introduction. In [1] Yano, Houh and Chen consider the structure called a \(\varphi(4, -2)\) structure, defined by a tensor field \(\varphi\) of type \((1,1)\) satisfying \(\varphi^4 - \varphi^2 = 0\) and they study the existence of this structure.

In this paper we study a \(\varphi(4, -2)\) structure of rank \(r = n - 1\) and the restriction of the structure \(\varphi\) on the hypersurface \(K\). In 3. we shall examine the relation between the almost product structure \(\Phi = 2\varphi^2 - 1\) and \(\varphi/K^*\).

1. Preliminaries. Let \(M^n\) be an \(n\)-dimensional differentiable manifold of class \(C^\infty\), and let the \(C^\infty(1,1)\) tensor fields \(f_1\) and \(f_2\) be given such that \(f_1^2 = 1\), \(f_2^2 = 0\). Then \(f_1\) is an almost product structure, and \(f_2\) is an almost tangent structure. Let a tensor field \(\varphi, \varphi \neq 0\) and \(\varphi \neq 1\), of type \((1,1)\) and of class \(C^\infty\) be given on \(M^n\) such that \(\varphi^4 - \varphi^2 = 0\) and rank \(\varphi = (\text{rank } \varphi^2 + \text{dim } M^n)/2 = r\).

Let \(1 = \varphi^2, \ m = 1 - \varphi^2\), then \(\varphi^1 = \varphi = \varphi^3, \ \varphi m = \varphi \varphi = \varphi - \varphi^3, \ \varphi^2 I = I^2 = 1, \ \varphi^2 m = m \varphi^2 = 0\).

Let \(\Phi = 1 - m = 2\varphi^2 - 1\). Then it is clear that \(\Phi\) defines on \(M^n\) an almost product structure if \(\varphi^2 \neq 1\). Let \(L\) and \(M\) be the distributions corresponding to \(1\) and \(m\) respectively. We assume that \(\varphi = \varphi/L\) is not the identity operator of \(L\). Then \(\varphi\) acts on \(L\) as an almost product structure operator and on \(M\) as an almost tangent structure operator. Moreover, \(\text{dim } M = 2(n - r)\) and \(\text{dim } L = 2r - n\). Such a structure \(\varphi\) is called a \(\varphi(4, -2)\) structure of rank \(r\).


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If the rank of $\varphi$ is maximal, $r = n$, the $\varphi(4, -2)$-structure is an almost product structure and if the rank of $\varphi$ is minimal, $2r = n$, the $\varphi(4, -2)$-structure is an almost tangent structure.

In [1] it has been proved that a necessary and sufficient condition for an $n$-dimensional manifold to admit a tensor field $\varphi$, $\varphi \neq 0$ and $\varphi \neq 1$ of type (1,1) defining a $\varphi(4, -2)$-structure is that the group of the tangent bundle of the manifold be reduced to the group $0(h) \times 0(2r - n - h) \times 0(n - r) \times 0(n - r)$ $h = \dim L_1$, $L_1$ being the subspace of $L$ corresponding to the eigen value $+1$ of $\varphi$.

With respect to the adapted frame the tensors $g_{ij}$ and $\varphi^i_j$ have the components

$$ g = \begin{bmatrix} E_h & 0 & 0 & 0 \\ 0 & E_{2r-n-h} & 0 & 0 \\ 0 & 0 & E_{n-r} & 0 \\ 0 & 0 & 0 & E_{n-r} \end{bmatrix}, \quad \varphi = \begin{bmatrix} E_h & 0 & 0 & 0 \\ 0 & -E_{2r-n-h} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E_{n-r} & 0 \end{bmatrix} $$

I. Sato [2] introduced and studied almost paracontact Riemannian manifold $V$ with the structure $(\psi, \xi, \eta, g)$ that is, an $n$-dimensional differentiable manifold with a tensor field $\psi$ of type (1,1), a positive definite Riemannian metric $g$, a vector field $\xi$ and a 1-form $\eta$ satisfying

1. $\psi^2 = I - \iota \xi$, $\psi \xi = 0$, $\eta \psi = 0$, $\eta(\xi) = 1$,

2. $\eta(\xi) = g(\xi, X)$, $\eta(\psi X, \psi Y) = g(X, Y) - \eta(X) \eta(Y)$, $X, Y \in \mathcal{X}(V)$

where $I$ is the identity and $\mathcal{X}(V)$ denotes the set of differentiable vector fields on $V$. Such a manifold is called an almost paracontact Riemannian manifold, and its structure an almost paracontact Riemannian structure. A structure which satisfies only condition (1) is called a Sato structure. The following theorem is proved in [4].

**Theorem 1.1.** The almost product structure $\Phi$ induces on a hypersurface the Sato structure $\psi$ in the following way

$$ \Phi B = B \psi \oplus (\eta \otimes N), \quad \Phi N = B \xi, $$

where $B$ is the differential of the immersion $i$ Kinto $\mathcal{M}^n$.

**Proof.** $\Phi B = B \psi \oplus (\eta \otimes N), \quad \Phi^2 B X = \Phi [B \psi \oplus (\eta \otimes N)] X$, $\Phi^2 B X = \Phi [B \psi X \oplus \eta(X) N], \quad B X = \Phi B (\psi X) + \eta(X) \Phi (N)$

$$ B X = [B \psi \oplus \eta \otimes N] (\psi X) + \eta(X) \Phi (N), $$$$ B X = [B \psi^2 (X) + \eta \psi(X) N] + \eta(X) \Phi (N) $$

$$ B X = B(X) - \eta(X) B \xi + 0 + \eta(X) B \xi, \quad B X = B X $$

and

$$ \Phi N = B \xi, \quad \Phi^2 N = \Phi B \xi, $$

$$ N = (B \psi \oplus (\eta \otimes N)) \xi, \quad N = B \psi \xi + \eta(\xi) N, \quad N = N. $$
2. The structure \((\bar{\psi}, \xi, \eta, \mathcal{g})\), on \(K\). We shall assume that rank \(\varphi = n - 1\). Then \(M\) is a 2-dimensional manifold. Let \(K\) be a hypersurface in \(M^n\) orthogonal on vector
\[
N = \begin{pmatrix}
0 \\
\vdots \\
0 \\
-1 \\
0
\end{pmatrix} \quad \text{in} \quad M^n.
\]

Let \(\bar{\varphi}, \bar{m}\) and \(\bar{g}\) be restrictions of the structure \(\varphi\) and tensors \(m\) and \(g\) on \(K\), and let
\[
\xi = \begin{pmatrix}
0 \\
\vdots \\
0 \\
1
\end{pmatrix} \quad n - 1 \quad \eta = (0, \ldots, 0, 1)_{n-1}.
\]
\(\bar{\varphi}, \bar{m}\) and \(\bar{g}\) have matrixes of the form
\[
\bar{\varphi} = \begin{bmatrix}
E_h & 0 & 0 \\
0 & -E_{2\tau-n-h} & 0 \\
0 & 0 & 0
\end{bmatrix} \quad \bar{m} = \begin{bmatrix}
0_h & 0 & 0 \\
0 & 0_{2\tau-n-h} & 0 \\
0 & 0 & 1
\end{bmatrix} \quad \bar{g} = \begin{bmatrix}
E_h & 0 & 0 \\
0 & E_{2\tau-n-h} & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

**Theorem 2.1.** \(\bar{\varphi}\) is a Sato structure.

**Proof.** Since \(\bar{\varphi}^2 = 1 - \bar{m}\), multiplying the corresponding matrices it is clear that \(\bar{m} = \xi \eta, \quad \bar{\varphi}^2 = I - \eta \otimes \xi, \quad \bar{\varphi}^3 = 0, \quad \bar{\varphi}^4 = 0, \quad \xi(\eta) = 1, \) and moreover:

**Theorem 2.2.** \((\bar{\varphi}, \xi, \eta, \mathcal{g})\) is an almost paracontact Riemannian structure on \(K\).

**Proof.** It is clear that \(\eta(X) = \mathcal{g}(\xi, X), \quad \mathcal{g}(\bar{\varphi}X, \bar{\varphi}Y) = \mathcal{g}(X, Y) - \eta(X)\eta(Y)\) which proves the theorem.

In Theorem 1.1. it is proved that an almost product structure induces on a hypersurface a Sato structure. From this and from Theorems 2.1 and 2.2, we obtain the following:

**Theorem 2.3.** The almost product structure \(\Phi = 2\varphi^2 - 1\) induces on \(K\) a structure Sato moreover an almost paracontact Riemannian structure.

3. Relation between \(\psi\) and the \((\bar{\varphi}, \xi, \eta, \mathcal{g})\) structure. We shall examine what conditions must be satisfied so that the structure \(\psi\) induced by \(\Phi = 2\varphi^2 - 1\) on \(K^*\) is equal to the structure \(\bar{\varphi}\).
Let $K^*$ be the subspace of $K$ whose vectors have the form

\[
x = \begin{bmatrix}
x_1 \\
\vdots \\
x_h \\
0_1 \\
\vdots \\
0_{2r-n-h} \\
z_1
\end{bmatrix}
\]

**Theorem 3.1.** The almost product structure $\Phi$ induces on $K^*$ the Sato structure $\varphi$.

**Proof.** We shall prove the relations $\Phi B = B\varphi \oplus (\eta \otimes N)$ and $\Phi N = B\xi$ on $K^*$. That $\Phi N = B\xi$ is clear using

\[
\Phi = \begin{bmatrix}
E_h & 0 & 0 & 0 \\
0 & E_{2r-n-h} & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]

To prove the relation $\Phi B = B\varphi \oplus (\eta \otimes N)$ on $K^*$, we shall prove $BX = \Phi B(\varphi X) + \eta(X)\Phi(N)$ for the vectors $X \in K^*$.

Let $X \in K$, we obtain

\[
BX = \begin{bmatrix}
x_1 \\
\vdots \\
x_h \\
y_1 \\
\vdots \\
y_{2r-n-h} \\
z_1 \\
0
\end{bmatrix}, \quad \Phi B(\varphi X) + \eta(X)\Phi(N) = \begin{bmatrix}
x_1 \\
\vdots \\
x_h \\
y_1 \\
\vdots \\
y_{2r-n-h} \\
z_1 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
x_1 \\
\vdots \\
x_h \\
y_1 \\
\vdots \\
y_{2r-n-h} \\
z_1 \\
0
\end{bmatrix}
\]

when $y_1 = 0, \ldots, y_{2r-n-h} = 0$. From this it is easy to see that $\Phi B = B\varphi \oplus (\eta \otimes N)$ only on the space $K^*$. This proves the Theorem.
The \((\eta, \xi, \psi, \varphi)\) structure on subspaces of the space with the \(\varphi(4, -2)\) structure

Since \(\varphi\) and \(\varphi\) satisfy the following on \(K^*\): \(\eta(X) = \varphi(\xi, X)\), \(g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)\), we have

**Theorem 3.2.** The almost product structure \(\varphi\) induces on \(K^*\) the almost paracontant Riemannian structure \((\varphi, \xi, \eta, \varphi)\).

**References**

[1] K. Yano, C. Hohn, B. Chen, *Structures defined by a tensor field \(\varphi\) of type \((1,1)\) satisfying \(\varphi^4 + \varphi^2 = 0\)*, Tensor, N. S. 23 (1972) 81–87.


Univerzitet u Novom Sadu
Fakultet tehničkih nauka
Institut za primenjene osnovne discipline
21000 Novi Sad
Jugoslavija

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