CONDITIONS FOR THE INTEGRABILITY OF SECOND ORDER
NONLINEAR DIFFERENTIAL EQUATION, II

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Abstract. Conditions for the integrability of second order nonlinear differential equation (0.1) are derived. The obtained result contains, as a special case a number of known results.

0. In this paper we continue the investigation of integrability of non-linear second order differential equations. We consider the following equation:

\[ y'' + P(y)y'^2 + Q(x,y)y' + R(x,y) = 0, \]

where \( P, Q, R \) are given functions.

Equations of this type were considered in a number of papers (see [1-16]). Only in Kamke’s collection [1] 103 equations of the form (0.1) are noted. Also, Painlevé [2] considered many equations of the same form.

We propose a method for solving equation (0.1) by reducing it to an equation of the form:

\[ Y'' + A(Y,x)Y' + B(Y,x) = 0, \]

which is considered in [5]. In [5] was proved that the equation (0.2) can be reduced to the autonomous equation

\[ d^2 z/dt^2 + f(z)dz/dt + g(z) = 0 \]

by means of transformations

\[ y = q(x)z(t) + r(x), \quad dt = p(x)dx, \]

if the following conditions are fulfilled:

\[ A(Y,x) = pf((Y-r)/q) - 2q'/q - p'/p, \]
\[ B(Y,x) = p^2 qg((Y-r)/q) - (q''/q + A(Y,x)q'/q)(Y-r) - (r'' + A(Y,x)r') \]

AMS Subject Classification (1980): Primary 34A05
\((r, q, r)\) are some functions depending of one variable, \(p\) is a differentiable function of \(x\); \(q, r\) are twice differentiable functions of \(x\).

In Section 1 we shall derive the conditions for the integrability of (0.1) by using the above result from [5]. Also, some remarks and examples are given.

A certain special class of equations of the form (0.1) is treated in Section 2. Some particular cases are given and compared with some known results from [7-9], [13-17].

1. Substituting, in (0.1)

\[ Y = F(y), \]

(1.1)

where

\[ F(y) = K \left( \int_{y_0}^{y} \exp\left( \int_{y_0}^{y} P(y) dy \right) dy + L \right) \]

(1.2)

\((K, L)\) are constants\) we obtain (0.2), where the functions \(A, B\) are given by:

\[ A(K \left( \int_{y_0}^{y} \exp\left( \int_{y_0}^{y} P(y) dy \right) dy + L \right), x) = Q(x, y), \]

(1.3)

\[ B(K \left( \int_{y_0}^{y} \exp\left( \int_{y_0}^{y} P(y) dy \right) dy + L \right), x) = K \exp\left( \int_{y_0}^{y} (P(y) dy R(x, y) \right) \]

This means that the equations (0.1) and (0.2) are equivalent.

Furthermore, using (0.3) — (0.5) and (1.1) — (1.3), we conclude that (0.1) can be reduced to the autonomous form if the functions \(Q, R\) are given by:

\[ Q(x, y) = pf\left( \frac{(F(y) - r)}{q} \right) - 2q'/q - p'/p, \]

\[ R(x, y) = p^2 q g\left( \frac{(F(y) - r)}{q} \right) F'(y) - \left( q''/q + Q(x, y) q'/q \right) (F(y) - r) - \left( r'' + Q(x, y) r' \right). \]

In this case under the transformation

\[ F(y) = q(x) z(t) + r(x), \quad dt = p(x) dx, \]

(1.5)

where \(F\) is given by (1.2), the equation (0.1) reduces to the autonomous equation (0.3).

Then we obtain that the general solution of (0.1) is given by:

\[ \int \left( U(F(y) - r)/q, C \right)^{-1} d((F(y) - r)/q) = \int p(x) dx + D, \]

(1.6)
\( (C, D) \) are arbitrary constants, \( u = U(z, C) \) is the general solution of the first order equation

\[
(1.7) \quad u(z)du/dz + f(z)u(z) + g(z) = 0
\]

**Remarks and examples.** Here are some remarks and examples related to the above result.

1° Equation

\[
(1.8) \quad y'' + P(y)y'^2 + a(x)y' + b(x)T(y) = 0,
\]

under the transformation \( (1.1) \) \( \rightarrow \) \( (1.2) \) reduces to the

\[
Y'' + a(x)Y' + b(x)S(Y) = 0
\]

where \( S(F(y)) = F'(y)T(y) \). If \( b(x) = k \exp(-2 \int a(x)dx)(k = \text{const}) \) then the above equation is integrable (see e.g. \([5, 10, 11]\)), so in this case, \( (1.8) \) is also integrable. This result is also obtained by J. D. Kečkić \([10]\) and by L. M. Berković and N. N. Rozov \([11]\).


3° Let \( r(z) = 0 \) and \( Q(x, y) = a(x) \). Then the following equation

\[
y'' + F''(y)F'(y)^{-1}y'^2 + a(x)y' + c \exp(-2 \int a \, dx)q^{-3}F'(y)^{-1}g(F(y)/q)\]

\[
-\frac{q''}{q} + a(x)\frac{q'}{q}F(y)F'(y)^{-1} = 0, \quad (c = \text{const})
\]

has the general solution

\[
(1.9) \quad \int (-2 \int g(F/q)d(F/q) + C)^{-1/2}d(F/q) = \int q^{-2} \exp(- \int a \, dx)dx + D
\]

\((C, D \) are arbitrary constants, \( F = F(y) \)).

In particular, the following equations

\[
y'' + (k - 1)y^{-1}y'^2 + a(x)y' + c \exp(-2 \int a \, dx)y^{1-k}q^{-3}g(y^k/q)\]

\[
-\frac{q''}{q} + a(x)\frac{q'}{q}y/k = 0,
\]

\[
y'' + ky^2 + a(x)y + c \exp(-2 \int a \, dx)q^{-3}e^{-ky}g(e^ky/q) - \frac{q''}{q} + a(x)q'/q/k = 0
\]

\[
y'' - y'^2/a(x)y' + c \exp(-2 \int a \, dx)yq^{-3}g\log y/q - \frac{q''}{q} + a(x)q'/q \log y/y = 0
\]
have the general solutions given by (1.9), where \( F(y) = y^k \), \( F(y) = e^{ky} \),
\( F(y) = \log y \), respectively. The above equations (first and third) appear in [12].

4° Equations
\[
y'' + F''(y) F'(y)^{-1} y' + a(x) y' + \sum_{i=1}^{m} b_i(x) F(y)^{n_i} F'(y)^{-1} = 0,
\]
\[
y'' + ky' + a(x) y' + \sum_{i=1}^{m} b_i(x) \exp((n_i - 1)ky)/k = 0,
\]
\[
y'' + (k-1)y'^{2} + a(x) y' + \sum_{i=1}^{m} b_i(x) y^{(n_i-1)k+1}/k = 0,
\]
where
\[
b_i(x) = c_i \exp(-2 \int a \, dx)(C_1 + C_2 \int \exp(- \int a \, dx) \, dx + C_3(\int \exp(- \int a \, px) \, dx)^2)
\]
\((C_1, C_2, C_3, c_i, n_i)\) are constants, \( i = 1, \ldots, m \), are integrable by quadratures.

The above equations are equivalent to
\[
Y'' + a(x) Y' + \sum_{i=1}^{m} b_i(X) Y^{n_i} = 0,
\]
which is also integrable if \( b_i \) have the form (1.10) (see [5]).

5° Let \( q(x) = 1 \), and \( Q(x, y) = a(x) \). Then the following equation
\[
y'' + F''(y) F'(y)^{-1} y'^{2} + a(x) y + c \exp(-2 \int a \, dx) F'(y)^{-1} g(F(y) - r) -
-(r'' + a(x) r') F'(y)^{-1} = 0, \quad (c = \text{const})
\]
has the general solution
\[
\int \left( -2 \int g(F - r) \, d(F - r) + C \right)^{-1/2} \, d(F - r) \quad \text{is integrable}
\]
\((C, D\) are arbitrary constants). 

6° Equations
\[
y'' + F''(y) F'(y)^{-1} y'^{2} + a(x) y' + \sum_{i=1}^{m} b_i(x) \exp(n_i F(y)) F'(y)^{-1} = 0,
\]
\[
y'' - y'^{2} + a(x) y' + \sum_{i=1}^{m} b_i(x) y^{n_i+1} = 0,
\]
where \( n_i(i = 1, \ldots, m) \) are constants and \( b_i \) are given by
\[
b_i(x) = c_i \exp(-2 \int a \, dx - n_i r(x)) \quad \text{for} \quad i = 1, \ldots, m
\]
\[
r(x) = C_1 + C_2 \int \exp(- \int a \, dx) \, dx + C_3(\int \exp(- \int a \, dx) \, dx)^2
\]
\((C_1, C_2, C_3 \text{ are constants})\), are integrable. This follows from the equation
\[
Y'' + a(x)Y' + \sum_{i=1}^{m} b_i(x) \exp(n_i Y) = 0,
\]
which is integrable if \(b_i\) is given by (1.12) (see [5]).

2. Let he equation (0.2) be of the form:
\[
y'' + a(x)y' + b(x)y + c(x)y'' = 0 \quad (n = \text{const}),
\]
\((a, b, c \text{ are given functions})\). Equation of the type (0.1) corresponding to (2.1) is:
\[
y'' + P(y)y' + a(x)y' + b(x)Q_1(y) + c(x)Q_2(y) = 0,
\]
where \(P, Q_1, Q_2\), satisfy the following system
\[
Q_1'(y) = 1 - P(y)Q_1(y), \quad Q_1(y)Q_2'(y) = (n - P(y)Q_1(y))Q_2(y).
\]

Equation (2.2) — (2.3) reduces to the (2.1) by the transformation
\[
Y = F(y) = (Q_2(y)/Q_1(y))^{1/(n-1)}.
\]

For the equation (2.1) the following result is known (see e.g. [7]):

Equation (2.1) has the solution \(Y = u(x)Z(v(x)/u(x))\), where \(u\) and \(v\) are linearly independent solutions of the linear equation:
\[
u'' + a(x)/u' + b(x)u = 0.
\]
with the Wronskian \(W = v'u - u'v\); \(Z\) is a solution of the following nonlinear equation
\[
d^2Z/dt^2 + h(t)Z^n = 0,
\]
where \(h\) is determined by
\[
h(v(x)/u(x)) = u(x)^{n+3}c(x)W^{-2}.
\]

Using the above we can formulate the following result for the equation (2.2)—(2.3):

Equation (2.2)—(2.3) has the solution
\[
F(y) = u(x)Z(v(x)/u(x)),
\]
where \(u\) and \(v\) are linearly independent solutions of (2.5); \(F\) is given by (2.4); \(Z\) satisfies the equation (2.6)—(2.7).

Remarks and examples. 1\(^a\) Using (2.4) we obtain that the equation (2.2)—(2.3) can be represented in the following form:
\[
y'' + F''(y)F'(y)^{-1}y'-a(x)y' + b(x)F(y)F'(y)^{-1} + c(x)F(y)^nF'(y)^{-1} = 0.
\]
2° Let \( h(t) = k(C_1 + C_2 t + C_3 t^2)^{-(n+3)/2}(k, C_1, C_2, C_3, = \text{const}) \). Then the
equation (2.6) is integrable by quadratures (see e.g. [7]) and from (2.7) we find the
function \( c \):

(2.9) \[ c(x) = kW^{2}(C_1 u^2 + C_2 uv + C_3 v^2)^{-(n+3)/2}. \]

In this case the equation (2.2) — (2.3) — (2.11) has the solution

(2.10) \[ F(y) = (C_1 u^2 + C_2 uv + C_3 v^2)^{1/2} U(\int (C_1 + C_2 v/u + C_3 v^2 /u^2)^{-1} d(v/u)), \]

where \( U(s) \) is given by

(2.11) \[ \int (C - (C_1 C_3 - C^2_2 /4) U^2 - 2k U^{n+1} /(n + 1))^{-1/2} dU = s + D \ (n \neq -1), \]

(2.11) \[ \int (C - (C_1 C_3 - C^2_2 /4) Y^2 - 2k \log U)^{-1/2} dU = s + D \ (n = -1). \]

(\( C, D \) are arbitrary constants, \( F \) is given by (2.4)).

3° Furthermore, let

(2.12) \[ c(x) = W^{2} u^{-n-3} \sum_{i=1}^{m} \left( a_{i}(v/u)^{b_{i}} \right)^{2} \sum_{j=1}^{m} A_{ij}(v/u)^{B_{ij}}, \]

where \( q = k(1 - n) - 2, \ A_{ij} = ka_{i}a_{j}b_{i}(1 - b_{i} + (1 - k)b_{i}) \), \( B_{ij} = b_{i} + b_{j} - 2, k, a_{i}, b_{i} \)
are constants \((i, j = 1, \ldots, m)\).

In this case the equation (2.2)—(2.3)—(2.12) has the solution:

(2.13) \[ F(y) = u \left( \sum_{i=1}^{m} a_{i}(v/u)^{b_{i}} \right)^{k} \]

where \( F \) is determined by (2.4).

4° In particular if \( C_1 = 1, C_2 = C_3 = 0 \), the equation (2.2) — (2.3) — (2.9)
becomes

(2.14) \[ y'' + P(y)y' + a(x)y' + b(x)Q_{1}(y) + k(W^{2} u^{-n-3} Q_{2}(y)) = 0 \]

(\( P, Q_1, Q_2 \) satisfy (2.3)) and has the general solution given by (2.10) — (2.11). with
\( C_1 = 1, C_2 = C_3 = 0 \).

In the special case \( n = -3 \) the equation (2.14) reduces to the well known
Herbst’s equation (see e.g. [3, pp. 62–64], [4, pp. 187–188] [6, 8, 9]). Also, from
the above we obtain the general solution of the Herbst’s equation.

If \( a(x) = 0, b(x) = b = \text{const}, \ c(x) = c = \text{const} \) and \( n = 3 \) equation (2.14) —
(2.3) reduces to the equation (2.5) from [15].

5° Let \( P(y) = (p - 1)/y \ (p \neq 0) \). Then the equation (2.2)
becomes

(2.15) \[ y'' + (p - 1) y^{-1} y' + a(x)y' + b(x)p^{-1} y = c(x)p^{-1} y^{(n-1)p+1} = 0. \]
In this case \( F(y) = y^p \). Some special cases of the equation (2.15) with \( c(x) \) given by (2.15) and \( k = 2/(n - 1) \), i.e. \( q = 0 \) are treated in [13, 14, 15]. These results can be obtained as a particular cases of the above. Also, equation (2.15) is treated in [17].

REFERENCES


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