GRAPH EQUATIONS FOR LINE GRAPHS 
AND $n$-th DISTANCE GRAPHS

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0. Introduction: In this paper we will investigate the graph equation

(1)  \[ L(G) = D_n(G), \]

$L(G)$ being a line graph of $G$, while $D_n(G)$ is the $n$-th distance graph of $G$, i.e., $D_n(G)$ is a graph having the same vertex set as $G$ with two vertices $u$ and $v$ being adjacent in $D_n(G)$ if the distance $d_G(u,v)$ between $u$ and $v$ in $G$ equals $n$. Note that there is a similar operation in literature, namely, the $n$-th path graph (see [1] for a definition).

If $n = 1$, equation (1) becomes a classical one, i.e. $L(G) = G$, which was solved in [2]. The case $n = 2$ is in fact, identical with the equation $L(G) = P_n(G)$, where $n$ denotes the $n$-th path graph (3). Accordingly, we will assume throughout this paper that $n > 2$.

All usual terminology in this paper is taken from (4). Here we quote some unusual terminology. We say that graph is even (odd), if all its cycles have even (odd) lengths. $G(v_1, \ldots, v_m)$ denotes an induced subgraph of $G$ obtained by taking the vertices $v_1, \ldots, v_m$ as its vertex set. If $H$ is an induced subgraph of $G$, we write $H \subseteq G$. In general, if $H$ is any subgraph of $G$ we write $H \leq G$. $H$ is a distance-preserving subgraph of $G$, if $H$, as a subgraph of $G$, satisfies $d_H(u,v) = d_G(u,v)$, for any pair $u,v$ of vertices from $H$. The graph $C_m(k_1, \ldots, k_m)$ is obtained as follows: we take a cycle $C_m$ of length $m$ (the vertices of the cycle are labeled from 1 to $m$) and append to the $i$-th vertex a path of length $k_i$.

We also make some general remarks concerning the equations of the type

(2)  \[ F(G) = G, \]

where $F$ is assumed to be the graph valued function which is additive with respect to union (of graphs) and also preserves the connectedness of components. Taking

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\( G = \bigcup G_i \) (\( G_i \) is connected) as a solution to (2), we easily get that for some permutation \( \pi F(G_i) = G_{\pi[i]} \) holds for each \( i \). Recalling the cycle structure of \( \pi \), we conclude that the set of components can be partitioned in such a way that each part contains just the components which can be obtained one from each other using the iterations of \( F \). In other words, after the appropriate relabeling of components, it follows that any part takes the form:

\[
G_1, \ldots, G_i, \ldots, G_p,
\]

where for each \( i = 1, \ldots, p \), \( G_i = F^{i-1}(G_1) \), while \( F^p(G_1) = G_1 \). The number of parts clearly equals the number of cycles of \( \pi \), while the number of graphs in each part coincides with the length of the corresponding cycle. Assuming the labeling from (3) we can say that

\[
G^* = \bigcup_{i=1}^p G_i
\]
is also a solution to (2). Any solution of this type (which cannot be split into some other ones) will be referred to as a fundamental solution. Namely, any other solution can be represented as the union of the fundamental ones. So, in order to solve the equations of the type (2), we only need to find its fundamental solutions. Any of the graphs \( G_1, \ldots, G_p \) appearing in (3) may be referred to as a generator of the corresponding fundamental solution. The integer \( p \) is assumed to be the period of any graph \( G_1, \ldots, G_p \), since it is the smallest integer such that for each \( i = 1, \ldots, p \), \( F^p(G_i) = G_i \) holds. The above conclusions can be summarized in the following proposition.

**Proposition 1.** Under the assumptions above, the equation (2) is equivalent to the following family of equations

\[
F^p(G) = G,
\]
p being a natural number, while \( G \) is assumed to be connected.

Clearly, the same argument applies to the equations \( f(G) = g(G) \), if \( f \) or \( g \) is invertible at least on the solution set of the equation.

**1. Main considerations:** From now on we shall assume that \( G \) denotes a possible solution to (1). If so, suppose that \( G \) is a fundamental solution as well. To find it, it is enough to find any of its generators, and their common period if possible. By \((m, n)\) we denote the greatest common divisor of \( m \) and \( n \).

**Lemma 1.** The cycle \( C_k \) is a component of \( G \) if and only if \( k > 2n \) and \((k, n) = 1\). Moreover, \( C_k \) is a fundamental solution as well.

**Proof.** Let \( d = (k, n) \). Then

\[
D_n(C_k) = \begin{cases} 
  kK_1 & k < 2n \\
  nK_2 & k = 2n \\
  dC_{k/d} & k > 2n 
\end{cases}
\]
Now the proof of the Lemma immediately follows. □

Because of this lemma, from now on, we will ignore the cycles as components of $G$.

**Lemma 2.** Every component of $G$ which is not a cycle is unicyclic and odd.

**Proof.** Clearly, the number of points $p(G)$ equals the number of edges $g(G)$. Thus, to prove the unicyclic part, it is sufficient to show that $G$ has no acyclic components. Recalling that for any graph $H$ we have $\kappa(L(H)) \leq \kappa(H) \leq \kappa(D_n(H))$, where $\kappa$ denotes the number of components of the corresponding graph, we obtain

$$\kappa(L(G)) = \kappa(G) = D_n(G).$$

Clearly, no component of $G$ is trivial. If any, say $G_i$, is a tree, then for $n$ even, $D_n(G_i)$ is disconnected, which contradicts (6); for $n$ odd, $D_n(G_i)$ is bipartite (and a line graph as well), implying that $D_n(G_i)$ is a path or a cycle which in turn implies the same for some components of $G$. This is again impossible by (6); or the assumptions.

Suppose now some component of $G$ is unicyclic but even. Discussing the parity of $n4$, we obtain contradictions analogous to those above. □

**Lemma 3.** The maximal vertex degree of $G$ is less than 4.

**Proof.** Suppose the contrary and let $v$ be a vertex of $G$ such that $\deg v \geq 4$. Then $K_4 \subseteq L(G)$ and also $K_4 \subseteq D_n(G)$. Hence we can find in $G$ four vertices $v_1, \ldots, v_4$ at a mutual distance which is exactly $n$. Consequently, there is a subgraph of $G$, say $H$, such that

(a) $H$ is distance preserving,

(b) $H$ contains the vertices $v_1, v_2, v_3$,

(c) $H$ is critical with respect to (a) and (b).

By a slight effort it follows that $H$ must be one of the graphs $H_1$ or $H_2$ (see Fig. 1), depending on whether paths between $v_1, v_2, v_3$ have any vertex in common.

![Fig. 1.](image)

Case 1: $H_1$ appears in $G$. We now easily get

$$d(v_i, u_i) = d(u_j, u_k) + (n - g)/2 \quad (i \neq j \neq k \neq i),$$

(7)
where \( g = d(u_1, u_2) + d(u_2, u_3) + d(u_3, u_1) \) is, in fact, the girth of the component
that contains \( H_1 \). Since \( G \) is odd, by Lemma 2, the same follows for \( n \). Hence \( H_1 \)
may appear only for \( n \) odd. Next, consider \( v_4 \) as well. Of course, it cannot be in
\( H_1 \). Let \( u_k \) be a vertex of \( H_1 \) such that \( d(v_4, u_k) \) is minimal. Clearly, \( d(v_4, u_k) =
d(v_4, u_4) + d(u_4, v_4) = n \), implying that all \( d(u_4, v_i) \) are of the same parity. Since
\( d(v_i, u_k) + d(u_k, v_j) = n \) for some \( i \), \( j \) we get a contradiction.

Case 2: \( H_2 \) appears in \( G \). Now, we have

\[
d(v_1, u) = n/2
\]

and this implies that \( n \) is even. Hence \( H_2 \) may appear in \( G \) only if \( n \) is even. Next,
as above, consider \( v_5 \) and note that it cannot be in \( H_2 \). Let \( u_x \) be a vertex of \( H_2 \)
such that \( d(v_4, u_x) \) is minimal. Now we easily get \( u_x = u \). Thus for all \( i = 1, \ldots, 4 \),
we have \( d(v_i, u) = n/2 \). Since \( u \) cannot be isolated in \( D_n(G) \) by (6), there is a
vertex of \( G \) at distance \( n \) from \( u \), and consequently, a vertex, say \( w \), such that
\( d(u, w) = n/2 + 1 \). Without loss in generality, let \( w \) be adjacent to \( v_4 \); also let \( w_1, w_2, w_3 \)
be the neighbors of \( v_1, v_2, v_3 \) such that \( d(u, w_1) = n/2 - 1 \) (see Fig. 2).

![Fig. 2.](image)

Of course, \( d(w, w_i) \leq n \). Assume first, \( d(w, w_i) = n \) for all \( i \). But then
\( D_n(G) (w, w_1, w_2, w_3) = K_{1, 3} \) (note that \( d(w_i, w_j) < n \)). Thus, say \( d(w, w_3) = k \),
where \( k < n \). On the other hand, \( d(v_3, v_4) \leq d(v_3, w_3) + d(w_3, w) + d(w, w_4) = k + 2 \),
and hence \( k \geq n - 2 \). Moreover, if \( v_3 \) or \( v_4 \) belongs to the shortest path between \( w \)
and \( w_3 \), then \( d(v_3, v_4) < n \). Now if \( k = n - 2 \), we can find in \( G \) two different paths
between \( v_3 \) and \( v_4 \), each being of length \( n \). But this implies that \( G \) contains an even
cycle, which contradicts Lemma 2. So assume \( k = n - 1 \). But then, since \( d(w, w_i) = n - 1 \)
must hold for all \( i \), or just for one \( i \), we have \( D(G) (w, v_1, v_2, v_3, v_4) = K_{5 - x} \)
or \( D_n(G) (w, v_1, v_2, v_3) = K_{1, 3} \), an obvious contradiction. \( \square \)

The next lemma easily follows from the proof of the previous one.

**Lemma 4.** Any triangle of \( D_n(G) \) originates from some distance-preserving
subgraph of \( G \) which is (depending on the parity of \( n \) equal either to \( H_1 \) or \( H_2 \) (see
Fig. 1)).

In the next lemma we assume that \( G_u \cdot H_v \) is a dot product of rooted graphs
obtained by identifying their roots.

**Lemma 5.** No component of \( G \) is equal to the graphs

\[
C_m(k, 0, \ldots, 0) \quad (k > 1), \quad \text{or}
\]

\[
C_3 \cdot T_u \quad (T_u \text{ is a rooted tree}),
\]
Proof. The graphs (9) are eliminated by direct inspection. To eliminate the graphs (10) observe first that \( D_n(C_3 \cdot T_u) = D_n(C_3 \cdot T_u - x) \), where \( x \) is an edge of \( C \) nonincident with \( u \) (note, \( u \) is identified with some vertex of \( C_3 \)). The rest of the proof is as for Lemma 2. □

In order to prove the next lemma assume for a moment that any component of \( G \) is different from the graph \( C_m (k, 0, \ldots, 0) \) where \( k \leq 1 \). Namely, \( k = 0 \) is already assumed, while for \( k = 1 \), the corresponding graph, as will be pointed later on, is a generator of some fundamental solution with a period equal one. Hence, for any component \( G_i \) of \( G \), \( D_n(G_i) \), since being equal to \( L(G_j) \) for some \( j \), has at least two triangles.

**Lemma 6.** \( G \) has no triangles.

**Proof.** Suppose \( G_i \) is a component containing a triangle.

Case 1: \( n \) is odd. By Lemma 4, the graph \( H_1 \) of Fig. 1 appears now in \( G_i \) as a distance-preserving subgraph. Moreover, at least two copies of \( H_1 \) must appear in \( G_i \) of them having at least a triangle in common. If there are just two copies of \( H_1 \) in \( G_i \), then \( D_n(G_i) \) contains only two triangles which have an edge in common, and this contradicts (10). If more than two copies of \( H_1 \) appear in \( G_i \), then \( D_n(G_i) \) contains either \( K_{1,3} \) or \( C_4 \) as an induced subgraph, an obvious contradiction.

Case 2: \( n \) is even. By Lemma 4 again, graph \( H_2 \) of Fig. 1 is now a distance-preserving subgraph of \( G_i \). Let \( H_v \) be a copy \( H_2 \) with a central vertex \( v \). Choose \( t \) to be a vertex of the triangle of \( G_i \), so that \( d(v, t) \) is minimal. If \( n/2 \leq d(v, t) \leq n-2 \) (or \( d(v, t) \geq 3n/2 \)), then \( C_4 \) (respectively \( K_{1,3} \)) appears in \( D_n(G_i) \), a contradiction. Note also that at most two subgraphs \( H'_v \) and \( H''_v \) may exist in \( G_i \); otherwise \( C_4 \) or \( K_{1,3} \) appears in \( D_n(G_i) \) again. Observe now two subgraphs \( H_v \) and \( H_w \) of \( G_i \) (\( v \neq w \)) both equal to \( H_2 \). Then \( H_{vw} \) (see Fig. 3) appears in \( G \) as a distance-preserving subgraph.

Now \( d(v, w) \leq n \), since otherwise \( K_{1,3} \subseteq D_n(G_i) \). If \( 1 \leq d(v, w) \leq n-2 \), we get contradictions as follows. If \( d(v, w) = 1 \), then \( 2(K_4 - x) \subseteq D_n(G_i) \). The latter implies that some component of \( G \) has at least two triangles. On the other hand, if \( 2 \leq d(v, w) \leq n-2 \), then \( C_4 \subseteq D_n(G_i) \). So assume \( d(v, w) = n-1 \) or \( n \). Also suppose \( s \) is a vertex of \( H_{vw} \) with degree (in \( G_i \)) equal to 3 (more than 3 is impossible by Lemma 3). If \( s \) belongs to \( v - v_i \) or \( w - w_i \) \((i = 1, 2)\) paths of \( H_{vw} \), (note that \( s \neq v, w \)), then \( K_{1,3} \subseteq D_n(G_i) \). Otherwise, if \( s \) belongs to \( v - w \).
paths of $H_v$, then $C_4 \subseteq D_n(G_i)$ holds always except for the following possibility: $d(v, w) = n - 1$ and there exist two vertices $s, t$ on the $v - w$ path of $H_v$ such that $d(v, s) = d(t, w) = n/2 - 1$, $d(s, t) = 1$ and the edge $st$ belongs to the triangle of $G_i$. But then $2(K_4 - x) \subseteq D_n(G_i)$, a contradiction as earlier. Thus $D_n(G_i)$ may have at most two triangles, which have an edge in common. The latter contradicts (10). □

**Lemma 7.** Any vertex of $G$ having degree equal to 3 belongs to a cycle.

**Proof.** Suppose the contrary and let $G_i$ be a component of $G$ that contradicts the Lemma. Among the vertices of degree 3 not belonging to $C$ ($C$ is a unique cycle of $G_i$) choose $v$ so that $d(v, C)$ is as large as possible. Next, let $v_1$, $v_2$, $v_3$ be the neighbors of $v$ among which $v$ is the closest to $C$. Clearly, $v_1$, $v_2$, $v_3$ are mutually nonadjacent in $D_n(G_i)$ and belong to the same component of $D_n(G)$ (note, $D_n(G_i)$ must be connected). Hence, there must exist in $D_n(G_i)$ a vertex, say $u_x$, adjacent to at least one vertex among $v_1$, $v_2$, $v_3$. If the position of $u_x$ is observed in $G_i$, it follows that $u_x$ must be adjacent to precisely two vertices among $v_1$, $v_2$, $v_3$. If $u_x$ is adjacent in $D_n(G_i)$ to $v_3$ and $v_1$ (or $v_2$), then, due to the maximality in the choice of $v$, $u_x$ has no more neighbors in $D_n(G_i)$. Next, if $u_x$ is adjacent in $D_n(G_i)$ to $v_1$ and $v_2$, then $u_x$, again, has no more neighbors; otherwise if $w$ is neighbor of $u_x$, then, to avoid the appearing of $K_{1,3}$ or $K_4 - x$ in $D_n(G_i)$ $w$ must be adjacent, besides $u_x$, either to $v_1$ or $v_2$, which is impossible. It is also forbidden that two vertices $u_x$ and $u_y$ are adjacent to the same pair of vertices among $v_1$, $v_2$, $v_3$; $C_4$ or $K_4 - x$ appears again in $D_n(G_i)$. Thus, $D_n(G_i)$ contains $C_6$ as a subgraph, which is clearly a contradiction. □

We now focus our attention on the girth of all components which are not cycles: By $C$ we denote the unique cycle of the component under consideration. If $v$ and $C$ are in the same component, then $v_c$ denotes the vertex of $C$ which is unique of course) for which $d(v, C)$ is minimal: In the following lemmas we will discuss the girth of the components of $G$.

**Lemma 8.** No component of $G$ has a girth less than $2n - 3$.

**Proof.** It is sufficient to show that $D_n(G_i)$ is disconnected whenever for some component $G_i$ its girth $g(G_i)$ is less than $2n - 3$. To end this, we first split the vertex set of $G$ into two disjoint classes $V_1 = \{v\mid 1 \leq d(v, C) \leq n - 1 - h\}$ and $V_2 = \{v\mid v \in C \text{ or } d(v, C) \geq n - h\}$, where $h = [g(G_i)/2]$. Turning to $D_n(G_i)$, it must contain an edge, say $x$, such that $x = v_1 v_2$ and $v_1 \in V_1$, $v_2 \in V_2$. If so, $d(v_2, C) \geq n - h$ and also $(v_1)_C \neq (v_2)_C$: otherwise, deg $v_2$ equals 1 in $D_n(G_i)$ and this fact does not ensure the connectedness of $D_n(G_i)$. So we can find on $C$ two vertices $u_1$ and $u_2$ such that $d(v_2, u_1) = d(v_2, u_2) = n$, which together with $v_1$ and $v_2$ induce $K_1, 3$ in $D_n(G_i)$, providing a contradiction. □

In the following series of lemmas we investigate the components of $G$ having girth $2n - 1$.

**Lemma 9.** Let $G_i$ be a component of $G$ with $g(G_i) = 2n - 1$. If $u$ belongs to $C$ and deg $u = 3$, there exists a vertex $v$ of $G$ such that $d(u, v) = n$ and $d(u, v_i) = [(n - 1)/2]$ or $n - 1$. 
Proof. Clearly, \( u \) cannot be isolated in \( D_n(G_i) \). Thus, for some vertex \( v \), we have \( d(u, v) = n \). But then, in order to avoid \( K_{1,3} \) in \( D_n(G_i) \), we must have \( d(u, v) = [(n - 1)/2] \) or \( n - 1 \). \( \square \)

In the next few lemmas, when there is no indication to the contrary, we assume that \( n \) is odd.

**Lemma 10:** The graphs\(^1\) of Fig. 4 cannot be the induced subgraphs of \( G \).

![Fig. 4.](image)

**Proof.** If the graphs above really appear in \( G \), then \( C_6 \) appears in \( D_n(G) \) as an induced subgraph. The latter contradicts Lemma 2. \( \square \)

**Lemma 11.** Let \( G_i \) be a component of \( G \) with \( g(G_i) = 2n - 1 \). Then there is a pair of vertices in \( G_i \) not lying on \( C \) such that \( d(u, v) = n \). In addition \( d(u_c, v_c) = [(n - 1)/2] \) holds.

**Proof.** We first note that any pair of vertices originating from \( C \) is nonadjacent in \( D_n(G_i) \). Since \( D_n(G_1) \) cannot be bipartite, there must exist a pair of vertices \( u \) and \( v \), which satisfy the Lemma. If \( d(u_c, v_c) \neq (n - 1)/2 \), then \( K_{1,3} \subseteq D_n(G_i) \), a contradiction. \( \square \)

**Lemma 12.** If \( G \) is a component of \( G \) with \( g(G) = 2n - 1 \), then \( d(v, C) < n \) for any vertex \( t \) of \( G \).

**Proof.** Suppose the contrary and consider a vertex \( u \) such that \( d(u, C) \geq n \). Then \( d(u, C) = n \) since otherwise \( K_{1,3} \subseteq D_n(G_i) \). Let \( u, v_1, v_2, v_3, v_4 \) be the vertices of \( G_i \) as shown in Fig. 5a.

![Fig. 5a](image) ![Fig. 5b](image)

\(^1\)The labelings in Fig. 4 stand for the distances between corresponding vertices.
By Lemma 11, all vertices of $C$ except $u_c$ and possibly $v_1, v_2, v_3, v_4$ are of degree 2. Next, by Lemma 10, $\deg v_1$ or $\deg v_4$ is equal to 2, while, by Lemma 11, either $v_1$ or $v_4$ is degree 3. If for some vertex $w$, $d(w, C) > 0$ and $w_c$ is equal to $v_2$ or $v_3$, then, by Lemma 10 or 11, $d(w, C) < (n + 1)/2$. Also, if $n > 3$, then $\deg v_2 = \deg v_3 = 3$ cannot hold: otherwise $K_{1,3} \subseteq D_n(G_i)$. If $n = 3$, it easily follows that $d(u, C)$ cannot be equal to $n$. So, assume $\deg v_1 = 3, \deg v_4 = 2$, while $\deg v_2 = 2$ (or 3) and $\deg v_3 = 3$ (or 2). Now, let $p_i, q, r, s$ be the vertices chosen such that: $d(p, C) = 1, p_i = v_1; d(q, C) = (n - 1)/2, q = u_c; d(r, C) = d(s, C) = 0, d(q, r) = d(q, s) = n$. Clearly, $p, q, r, s$ induce in $D_n(G_i)$ a graph equal to $K_3 \cdot K_2$ (see Fig. 5b) and also $r$ and $s$ have no more neighbors in $D_n(G_i)$, as follows from Lemmas 10 and 11. The detail from Fig. 5b cannot appear in a line graph of any component of $G$ under the restriction registered thus far. □

**Lemma 13.** All components of $G$ having girth $2n - 1$ have exactly $4n - 2$ vertices.

**Proof.** Assume that $G_1, \ldots, G_p$ is a sequence of components for which

$$D_n(G_i) = L(G_{i+1}) \quad (i = 1, \ldots, p; G_{p+1} = G_1)$$

hold. Now it is easy to get that, if $g(G_i) = 2n - 1$ for some $i$, the same holds for each $i = 1, \ldots, p$. Next, for each component $G_i$, let $a_i, b_i$ and $c_i$ denote respectively the number of vertices on the cycle, the number of vertices outside the cycle and the number of vertices of degree 3 (or 1 as well). Also let $d_i$ denote the number of triangles in $D_n(G_i)$. By a simple argument, we have:

$$q(D_n(G_i)) = 2a_i + d_i, \quad q(L(G_{i+1})) = a_{i+1} + b_{i+1} + c_{i+1}.$$ 

From (11) and (12), since $a_i = a_{i+1} (= 2n - 1)$ and $c_{i+1} = d_i$ (= the number of triangles on each component side of (11)), we get $a_i = b_{i+1}$. Thus, $a_i + b_i = 4n - 2$ for each $i = 1, \ldots, p$. □

The next lemma is a direct consequence of Lemma 11.

**Lemma 14.** Let $G_i$ be a component of $G$ with $g(G_i) = 2n - 1$. If $u$ is a vertex of $G_i$ such that $d(u, C) = l > 0$ and $v$ is a vertex of $C$ satisfying $n - l - 1 \leq d(v, u_c) \leq n - 2$, then $\deg v = 2$, except possibly when $d(v, u_c) = (n - 1)/2$ (or $n/2 - 1$ if $n$ is even).

**Lemma 15.** Let $G_i$ be a component of $G$ with $g(G_i) = 2n - 1$. Suppose $u$ and $v$ are the vertices of $G_i$ such that $d(u, v) = n$, provided none of them is on $C$. Also, suppose $w$ is a vertex of $G_i$ not on $C$ such that $w_c$ lies between $u_c$ and $v_c$ on the shorter part of $C$. Then $d(w, C) \leq (n - 3)/2$.

**Proof.** If $d(w, u)$ or $d(w, v)$ is greater or equal to $n$, then $K_{1,3} \subseteq D_n(G_i)$. Therefore we have

$$d(u, u_c) + d(u_c, w_c) + d(w_c, w) \leq n - 1, \quad d(v, v_c) + d(v_c, w_c) + d(w_c, w) \leq n - 1$$

which implies $d(w, w_c) \leq (n - 2)/2$. □
Lemma 16. If $G_j$ is a component of $G$ with $g(G_j) = 2n - 1$, then whenever $n > 3$, the vertices of degree 3 in $G_j$ are nonadjacent.

Proof. Suppose the contrary. Since $L(G_j) = D_n(G_i)$ for some $i$, we have $K_3 \cdot K_3 \subseteq D_n(G_i)$. The latter implies that in $G_i$ there exists a vertex, say $v$, and four other vertices at distance $n$ from $v$. Using the same arguments as with Lemma 13 we get $g(G_i) = 2n - 1$. If $v$ is on the cycle $C$, then the four vertices mentioned are out of it. As also required, suppose that two of them, say $u$ and $w$, are at distance $n$. If so, paths of length $n$ between vertices $u$, $v$, and $w$ must be disjoint. Then we easily get $d(u_c, w_c) = (n + 1)/2$, a contradiction by Lemma 11. So, by using the foregoing lemmas, it follows that $G_i$ contains as an induced subgraph the graph of Fig. 6. In addition we have: $d(v, u) = d(v, w) = n$, $d(v_c, u_c) = d(v, w_c) = (n - 1)/2$, $d(v_c, t_1) = d(v_c, t_2) = n - 1$. Now, by Lemma 14, there are no vertices of degree 3 between $t_1$ and $t_3$, $t_2$ and $t_4$ (shorter parts of $C$ are assumed). According to the same lemma some vertices closer to $u_c$ (or $w_c$) depending on the length $d(w, w_c)$ (of $u, u_c$) are of degree 2. Next, assume there is a vertex $x$ between $u_c$ and $t_5$ (or $w_c$ and $t_6$) such that $deg x = 3$. By Lemma 9, there must exist a vertex $y$ such that $d(x, y) = n$ and $d(u, y) = (n - 1)/2$ or $n - 1$. If $d(x, y_c) = (n - 1)/2$, then $y_c$ falls between $u_c$ and $w_c$. Since $d(y, y_c) = (n - 1)/2$, this contradicts Lemma 15. If $d(x, y_c) = n - 1$, then $deg y_c = 2$, as already observed. Thus, besides $u_c$, $v_c$, $w_c$ only $t_1$, ..., $t_6$ could have degrees equal to 3.

For convenience, let $l(x) = \max_{y \in Y_x} d(y, C)$, where $Y_x = \{y | y_c = x\}$ (clearly, $x$ is a vertex of $C$).

We now make the following observations:

(14) $l(v) \leq (n - 1)/2$ (see Lemma 10);
(15) $l(t_1)$ or $l(t_5) = 0$, $l(t_2)$ or $l(t_6) = 0$, $l(t_3)$ or $l(t_4) = 0$ (see Lemma 11);
(16) $l(t_3) + l(u_c) + (t_4) = (n - 1)/2$ (see (14) if $l(t_3) = l(t_1)$, or (15) and Lemma 11 otherwise).

We now discuss conditions which ensure that $t_5$ and $t_6$ are not isolated in $D_n(G_i)$, i.e., we look for the vertices $x$ and $y$ such that $d(t_5, x)$ and $d(t_6, y)$ are equal to $n$. 

![Fig. 6.](image)
Case 1: $l(u_c) \geq (n+3)/2$. Now, $l(w_c) \leq (n-1)/2$ and $l(t_1) = 0$ by Lemma 10; $l(t_2) = l(t_3) = l(t_4) = 0$ by Lemma 11; $l(t_3) + l(v_c) + l(t_4) \leq (n-1)/2$ is just (16). Thus, since $l(u_c) \leq n-1$ (see Lemma 12) we get

\[ \sum_{x \in C} l(x) \leq 2n - 2, \]

which contradicts Lemma 13.

Case 2: $l(u_c), l(w_c) = (n-1)/2$. Now, $l(t_1) = l(t_2) = l(t_3) = l(t_4) = 0$, by Lemma 11. Observing (16) as well, we get contradiction by (17).

Case 3: $l(t_1), l(t_2) \geq 1$. Now, by (15), $l(t_5) = l(t_6) = 0$; $l(t_1) + l(w_c) \leq (n-1)/2$, $l(t_2) + l(u_c) \leq (n-3)/2$ by Lemma 11. If (16) is observed again, we get the same contradiction as in the previous case.

Case 4: $l(u_c) \geq (n-1)/2$, $l(t_2) \geq 1$. This possibility by itself contradicts Lemma 11.

So $t_5$ or $t_6$ is isolated in $D_n(G_i)$, a contradiction. \(\Box\)

**Lemma 17.** If $n$ is odd and $n > 3$, no component of $G$ has the girth equal to $2n - 1$.

**Proof.** Assume $G_i$ is a component of $G$ for which $g(G_i) = 2n - 1$. Since $D_n(G_i)$ cannot be bipartite there are two vertices in $G_i$, say $u$ and $v$, such that $d(u, v) = n$, and $m$ addition, neither $u$ nor $v$ belongs to $C$. Now, $G_i$ contains as an induced subgraph a graph exactly equal to the graph of Fig. 6 with one slight modification; namely, all vertices $x$ for which $x_c = w_c$ may be ignored; the rest is the same. As in the previous lemma, we first conclude that there are no vertices of degree 3 between $t_2$ and $t_4$ (the shorter part of $C$ is assumed). Now suppose $x$ is a vertex of degree 3 lying on the shorter part of $C$ between $u_c$ and $v_c$. Since $x$ cannot be isolated in $D_n(G_i)$ there is a vertex $y$ in $G_i$ such that $d(x, y) = n$. By Lemma 9, $d(x, y_c) = (n-1)/2$ or $n-1$. According to Lemmas 9, 11 and 16, any position of $y_c$ gives a contradiction. Thus, all vertices between $u_c$ and $v_c$ (on the shorter part of $C$) are of degree 2. So we have to examine whether any vertex of degree 3 can exist between $t_1$ and $u_c$, or $w_c$ and $v_c$ (in both cases shorter parts of $C$ are assumed). Considering Lemmas 9 and 16 as well, the only possibility for the existence of such vertices is that they appear in pairs so that their mutual distance is $n - 1$. Hence, one is between $t_1$ and $t_3$, while the other is between $w_c$ and $v_c$. Denote these vertices by $x$ and $y$. Using Lemma 11, we have:

\[ l(x) + d(x, u_c) + d(u_c, v_c) + l(v_c) \leq n-1, \quad l(y) + d(y, v_c) + d(v_c, u_c) + l(u_c) \leq n-1. \]

By adding these relations we get an obvious contradiction. Thus there are no vertices of degree 3 in the corresponding parts of $C$.

Till now, we have proven that besides $u_c$ and $v_c$, only $w_c$, $t_1$, $t_2$, $t_4$ possibly have degrees equal to 3. The rest of the proof runs in the same way as the corresponding part of the proof of the preceding lemma. \(\Box\)

The next lemma stems from the lemma above as a direct consequence.
Lemma 18. If $n = 3$ and if $G$ is a component of $G$ with $g(G_i) = 2n - 1$ ($= 5$), then $G_i = C_5(1, 1, 1, 1, 1)$; $G_i$, as a generator of some fundamental solution, has a period equal to 1.

We conclude consideration of the components of girth $2n - 1$ by letting $n$ be even.

Lemma 19. If $n$ is even, $G$ has no components of girth $2n - 1$.

Proof. Suppose the contrary and let $G_i$ be a component having the minimal number of vertices of degree 3. Since $D_n(G_i)$ must have at least one triangle, using Lemma 4, we get that the graph of Fig. 7 appears now in $G_i$ as an induced subgraph. Also we must have $d(v, v_c) = n/2$, $d(v, x) = n$, $d(v_c, u_c) = d(v_c, w_c) = n/2 - 1$ and $d(v_c, t_2) = d(v_c, t_2)n - 1$.

Using Lemma 14 (it holds for $n$ even as well), it follows that all vertices between $t_1$ and $u_c$, $t_2$ and $w_c$ (in both cases shorter parts of $C$ are assumed) have degrees equal to 2. By Lemma 9, there exists a vertex, say $y$, such that $d(x_c, y) = n$ while $d(x_c, y_c) = n/2 - 1$ or $n - 1$. Since $y$ is not on $C$ we must have $d(x_c, y_c) = n/2 - 1$, to avoid the forbidden parts of $C$. Also assume, $x_c \neq u_c$. Then, since $y_c$ must be between $v_c$ and $w_c$ (on the shorter part of $C$ and since $d(y, C) = n/2 + 1$, we easily get $K_4 - x \subseteq D_n(G_i)$, which contradicts Lemma 6. Moreover, by the same argument it follows that all vertices of $C$, except $v_c$ and possibly $u_c$, $w_c$, $t_1$, $t_2$, $t_3$, $t_4$ are of degree 2. In particular, $x_c$ coincides with $u_c$ (or $w_c$) in which case $d(x, C) = 1$, or $t_3$ (or $t_4$) in which case $d(x, C) = n/2 - 1$. The following facts can be easily verified:

![Fig. 7](image)

(18) $l(u_c)$ or $l(w_c) = 0$;

(19) $l(t_3)$ or $l(t_4) \geq n/2 - 1$, implies $l(u_c) = 1(w_c) = 0$;

(20) $l(t_3)$ or $l(t_4) < n/2 - 1$.

From (18)-(20), it follows that each vertex of $G_i$ at distance $n/2$ from the cycle contributes to the appearance of at most one triangle in $D_n(G_i)$. On the other hand $D_n(G_i) = L(G_j)$, where $G_j$, as can be easily deduced, is a component with the same girth as $G_i$. Due to the minimally restriction posed in the choice of $G_i$, it follows that each vertex of degree 3 in $G$ contributes to the appearance of just one triangle of $D_n(G_i)$. Thus, for each vertex $x$ of $C$, either $l(x) = 0$
or \( l(x) \geq n/2 \). Suppose now \( l(t_3) \geq n/2 \). Then \( l(t_1) = 0 \) (\( t_3 \) and its appended path is considered similarly as \( v_c \) and path \( v_c - v \)). Applying (19), it follows that, besides \( l(t_3) \) and \( l(v_c) \), only \( l(t_2) \) may be different from 0. In this case we can easily show, by constructing \( D_n(G_i) \), that \( C_{2n} \subseteq D_n(G_i) \), which contradicts Lemma 2. If \( l(u_c) \geq n/2 \), then by arguments similar to those above; it follows that only \( u_c \) and \( v_c \) can have degrees equal to 2. Thus, \( G_i \) consists of a cycle \( C \) and at most two hanging paths of length not less than \( n/2 \) which meet the cycle \( C \) in vertices at distances 1 or \( n/2 - 1 \). Now, except for \( n = 4 \), we could always find \( K_{1,3} \subseteq D_n(G_i) \).

By treating \( n = 4 \) separately, which is by no means a problem, we finish the proof. \( \square \)

**Lemma 20.** The components of \( G \) which are not cycles and which have their girth greater than \( 2n - 1 \) possibly exist if their girth is equal \( 3n - 2 \) or \( 3n \).

**Proof.** Suppose \( G_i \) is a component of \( G \) with \( g(G_i) > 2n - 1 \) and \( g(G_i) \neq 3n - 2, 3n \). Since \( G_i \) is not a cycle we can find four vertices \( v_1, v_2, v_3 \) of \( G_i \) such that \( d(v_1, v_2) = n \) and \( d(v_1, v_3) \neq n \) (note, one vertex out of \( v_1, v_2, v_3 \) is at distance 1 from \( C \). The latter implies \( K_{1,3} \subseteq D_n(G_i) \), a contradiction. \( \square \)

**Lemma 21.** \( G \) has no components of girth \( 3n \).

**Proof.** Suppose the contrary. Then, there exists a component, say \( G_i \), equal to \( C_{3n} (k_1, \ldots, k_{3n}) \) where not all \( k \)'s are equal to 0. If for some \( s \) \( k_s > \lfloor n/2 \rfloor \), then we easily get \( C + 6 \subseteq D_n(G_i) \), a contradiction by Lemma 2. Thus \( k_s \leq \lfloor n/2 \rfloor \) for any \( s \). Let \( U = \{u_1, \ldots, u_{3n}\} \) be the vertex set of \( C \) (\( u_s \) and \( u_{s+1} \) are adjacent), while \( V = \{v_1, \ldots, v_m\} \) are the remaining vertices of \( G_i \). Then each triplet of vertices \( u_s, u_{s+1}, u_{s+2n} \) forms a triangle in \( D_n(G_i) \), denoted by \( T_s \). Any vertex \( v_1 \), if regarded in \( D_n(G_i) \), is adjacent to just two vertices from \( U \), each of them belonging to different triangles (note that \( n \) is odd, while \( k_s \leq \lfloor n/2 \rfloor \)). Vertices from \( V \) cannot be adjacent in \( D_n(G_i) \), since otherwise we have \( K_{1,3} \subseteq D_n(G_i) \). Similarly, each vertex \( u_s \) may be adjacent to at most one vertex \( v_1 \). Next, let \( D_n(G_i) = L(G_j) \) for some \( j \). Since \( G_j \) is unicyclic and trianglefree, there must exist in \( D_n(G_i) \) a cycle of length greater then 3. Then, in \( G_i \), the following sequence of vertices (indices are ignored) corresponds to the cycle mentioned: \( uuuvv \ldots v \) (the first and the last member of the sequence correspond to the same vertex). If the number of occurrences of \( u \) in pairs is less than \( n \), then the vertices of some triangle \( T_s \) are not taken into account in the sequence above. Therefore it follows that \( L(G_j) \) contains two disjoint cycles, one of which is a triangle. In turn, this implies that \( G_j \) contains a vertex of degree 3 outside a cycle, a contradiction by Lemma 7. So we immediately get \( G_j = S \), where for brevity we put \( S = C_{3n} (1, 0, 0, 1, 0, 0, \ldots, 1, 0, 0) \). Since \( D_n(G_i) = L(G_j) \) implies \( G_j = S \), then we must have \( G_i = S \). Now, if \( n \equiv 1 \pmod{3} \) (or \( n \equiv 2 \pmod{3} \)), we get that some vertex from \( U \) has two neighbors in \( V \) (two vertices from \( V \) are adjacent). Thus, \( n \equiv 0 \pmod{3} \). If so, observe the vertices \( u_1, u_{n+1}, u_{2n+1} \) chosen so that their degrees are 3. These vertices induce in \( D_n(G_i) \) an isolated triangle, a contradiction. \( \square \)

Next we will investigate only the components of \( G \) with the girth \( 3n - 2 \).
Lemma 22. If $G_i$ is a component of $G$ with $g(G_i) = 3n - 2$, then $G_i = C_{3n-2}(k_1, \ldots, k_{3n-2})$, where $k_s \leq 1$.

Proof. Clearly, we only need to prove that $k_s \leq 1$ for all $s$. If $k_s > 1$ for some $s$, then $D_n(G_i)$ contains, among others, two cycles none of which is a triangle. This is a contradiction since all components of $G$ are unicyclic. □

Now, let $\mathcal{H} = \{H \mid H = C_{3n-2}(k_1, \ldots, k_{3n-2}), \text{ where } k_s \leq 1\}$. If $H \in \mathcal{H}$, let $W(H) = \{w \mid w \text{ is a vertex of } H \text{ with } \deg w = 3\}$. Next, let $\varphi^1 = L^{-1} \circ D_n$ provided that $L^{-1}$ does not produce isolated vertices.

Lemma 23. Let $H \in \mathcal{H}$. Then $\varphi(H) \in \mathcal{H}$, if the distance between any two vertices from $W(H)$ is not equal to $n - 2$.

Proof. We first note that $d(w_1, w_2) \neq n - 2$ for any $w_1, w_2 \in W(H)$, since otherwise $D_n(G_i)$ has at least two cycles none of which is a triangle. On the other hand, if $d(w_1, w_2) \neq n - 2$ for all vertex pairs from $W(M)$, we have $\varphi(H) \in \mathcal{H}$. □

Clearly, $H \in \mathcal{H}$ generates a fundamental solution to (1), if and only if $\varphi^p(H) \in \mathcal{H}$ for every $p > 0$. To end this, we first examine the effect of changing the distance between any two vertices from $W(H)$, going from $H$ to $\varphi(H)$.

Lemma 24. Suppose $u$ and $v$ are vertices of a cycle $C$ of length $3n - 2$ with $n$ odd. We then have $d_{D_n[C]}(u, v) = f(d_c(u, v))$ where $f$ is the following function:

$$
f(x) = \begin{cases} 
3x/2, & x \text{ is even and } 1 \leq x \leq n - 1; \\
3(n - x)/2 - 1, & x \text{ is odd and } 1 \leq x \leq n - 1; \\
3(n - x)/2 - 2, & x \text{ is even and } n + 1 \leq x \leq 3(n - 1)/2; \\
3(n - x)/2 + 1, & x \text{ is odd and } n + 1 \leq x \leq 3(n - 1)/2. 
\end{cases}
$$

Proof. Clearly, $d_{D_n[C]}(u, v) = \min(p, 3n - 2 - p)$, where $p$ is the smallest nonnegative integer such that $p + x + y = 3n - 2$ (for $q \geq 0$) and $x = d_c(u, v)$. Thus, we have $p = \min(3q - (2q - x)/n)$. Since $r = (2q - x)/n$ is an integer, we get $q = (nr + x)/2$, implying $p = \min(3(nr + x)/2 - r)$, where $r$ is an integer not less than $-x/n$. Next, if $x$ is even, then $r = 0$, while for $x$ being odd $r = \pm 1$, depending on the ratio of $x$ and $n$. So (21) easily follows. □

Assume now $H \in \mathcal{H}$ implies $\varphi(H) \in \mathcal{H}$, if so, define a mapping from $W(H)$ onto $W(\varphi(H))$ as follows: to each $w \in W(H)$, there corresponds a $w' \in W(\varphi(H))$ such that whenever a hanging edge at $w$ is deleted (which sets $H$ to $H^*$), then a hanging edge at $w'$, if deleted, gives $\varphi(H^*)$.

Lemma 25. Under the above assumptions, if $w_1, w_2 \in W(H)$, then

$$d_{\varphi(H)}(w_1, w_2) = f(d_H(w_1, w_2)),$$

where $f$ is given by (21).

Proof. Without loss in generality, let $W(H) = \{w_1, w_2\}$. To make it easier, observe Fig. 8, where $d(w_1, a_s) = d(w_2, b_s) = n - 1$ ($s = 1, 2$), while $c_2, c_2$ and $d_1, d_2$ replace the $d$’s and $b$’s in order to avoid the effects of permuting their indices. Following Fig. 8, we get: $d_{\varphi(H)}(w_1, w_2) = d_{D_n[H]}(c_s d_s)$ ($s = 1, 2$), while on the other hand $d_{\varphi(H)}(c_s, d_s) = f(d_H(u_t, b_t)) = f(d_H(w_1, w_2))$ ($s, t = 1, 2$). Now the Lemma easily follows. □
2. Main result. Collecting all the conclusions proven thus far in the lemmas, we arrive at our main result.

**Theorem.** The graph equation $L(G) = D_n(G)$ has, as the generators of the fundamental solutions, just the following graphs:

1° $H = C_k$, for $k > 2n$ and $(k,n) = 1$; the period of this graph is 1.

2° $H = C_5(1,1,1,1)$ only for $n = 3$; the period of this graph is 1.

3° $H = C_{3n-2}(k_1, \ldots , k_{3n-2})$ provided that:

a) $n$ is odd;

b) $k_i \leq 1$ for all $i = 1, \ldots , 3n - 2$;

c) if $u_i, u_j$ ($u_i \neq u_j$) are the vertices of the cycle for which $k_i, k_j \neq 0$, then $d(u_i, u_j) \not\in \{fp(n - 2) | p \geq 0\}$, where $f$ is in fact the permutation given by (21), or in other words $d(u_i, u_j)$ does not belong to the cycle of $f$ that contains $n - 2$.

The period of $H$ in the case 3 is an open question.

*Remark.* It follows from the theorem above that we can find a general solution for any particular $n$. The only inconvenience is that for an arbitrary $n$ we don’t know the period of some generator, i.e., we don’t know in advance the number of components of some fundamental solution.

In order to illustrate this theorem, we deduce the solution for $n = 3$. In this case the general solution consists of the components of the following three types: $C_k (k > 6$ and $k \not\equiv 0$ (mod $3$)), $C_5 (1,1,1,1,1)$ and $C_7 (1,0,0,0,0,0,0)$.

**REFERENCES**


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