A FIXED POINT THEOREM IN A REFLEXIVE BANACH SPACE

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In [1] the following theorem is proved:

**Theorem A.** Let $B$ a reflexive Banach space, $K$ a nonempty bounded closed and convex subset of $B$ and $T : K \to K$ a mapping satisfying the following conditions:

\[
\|Tx - Ty\| \leq \max\{\|x - Tx\|, \|y - Ty\|, (\|x - Ty\| + \|y - Tx\|)/3, \\
(\|x - y\| + \|x - Tx\| + \|yTy\|)/3\}, \quad x, y \in K
\]

and

\[
\sup_{z \in D} \|z - Tz\| \leq \delta(D)/2,
\]

where $D$ is any nonempty closed convex subset of $K$ which is mapped into itself by $T$ and $\delta(D) = \sup_{z, y \in D} \|x - y\|$ the diameter of $D$. Then $T$ has a unique fixed point in $K$.

In the present note we shall prove a theorem which is certain generalization of Theorem A, and its proof is simpler than that of Theorem A. Namely, we have the following:

**Theorem 1.** Let $B$ a reflexive Banach space, $K$ a nonempty bounded closed and convex subset of $B$ and $T : K \to K$ a mapping satisfying the following conditions:

\[
\|Tx - Ty\| \leq \max\{\|x - Tx\|, \|y - Ty\|, a\|x - Ty\| + b\|y - Tx\|, \\
(\|x - y\| + \|x - Tx\| + \|y - Ty\|)/3, \\
x, y \in K, \quad a \geq 0, b \geq 0, a + b < 1
\]

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and
\[
\sup_{z \in D} \|z - Tz\| \leq r\delta(D), \quad 0 \leq r = r(D) < 1,
\]
where \(D\) and \(\delta(D)\) have the same meaning as in Theorem A. Then \(T\) has a unique fixed point in \(K\).

Proof. Let \(\mathcal{F}\) denote the family of all nonempty bounded closed convex subsets of \(K\), which are mapped by \(T\) into itself. \(\mathcal{F}\) is nonempty since \(K \in \mathcal{F}\). If \(\|F_0\\|\) is any nonincreasing sequence in \(\mathcal{F}\), then by the well known result [2] of Smulian, \(F = \bigcap_i F_0\) is in \(\mathcal{F}\). Now, by Zorn’s lemma it follows that \(\mathcal{F}\) has a minimal element. If \(C\) is such a minimal element of \(\mathcal{F}\), we shall prove that \(C\) contains only one point, i.e., that \(T\) has a fixed point in \(K\). Supposing that \(C\) contains more than one element we obtain
\[
\sup_{x,y \in C} \|x - y\| = \delta(C) > 0
\]
Since \(T(C) \subseteq C\), for any \(x, y \in C\) we have, by (1), (2) and (3)
\[
\|T x - T y\| \leq \max\{r\delta(C), (a + b)\delta(C), (\delta(C) + 2r\delta(C))/3\}
\]
Putting \(\overline{r} = \max\{a + b, (1 + 2r)/3\} < 1\) we have
\[
\|T x - T y\| \leq \{r\delta(C), (\overline{r} < 1)\) for each \(x, y \in C\).
\]
If by \(\text{co} D\) we denote the convex hull of \(D\), and by \(\text{co} D\) the closed convex hull of \(D\) we have
\[
\text{co} T(C) \subseteq \text{co} C = \overline{C} = C
\]
because \(C \in \mathcal{F}\), \(T(C) \subseteq C\) and \(C\) is closed and convex. Therefore
\[
T(\text{co} T(C)) \subseteq T(C) \subseteq \text{co} T(C) \subseteq \text{co} T(C).
\]
Since \(C\) is a minimal element of \(\mathcal{F}\), by (5) and (6) we have \(\text{co} T(C) = C\). Let \(\overline{x}, \overline{y} \in \text{co} T(C)\). Then we can write
\[
\overline{x} = \sum_{i=1}^{n} a_i T x_i, \quad a_i \geq 0 (i = 1, \ldots, n), \quad \sum_{i=1}^{n} a_i = 1, \quad x_i \in C
\]
\[
\overline{y} = \sum_{j=1}^{m} b_j T y_j, \quad b_j \geq 0 (j = 1, \ldots, m), \quad \sum_{j=1}^{m} b_j = 1, \quad y_j \in C
\]
Now, by (4), and \(\sum_{i,j} a_i b_j = 1\)
\[
\|\overline{x} - \overline{y}\| = \left\| \sum_{i=1}^{n} a_i T x_i - \sum_{j=1}^{m} b_j T y_j \right\| = \left\| \sum_{i,j} a_i b_j T x_i - \sum_{i,j} a_i b_j T y_j \right\| =
\]
\[
\leq \left\| \sum_{i,j} a_i b_j (T x_i - T y_j) \right\| \leq \sum_{i,j} a_i b_j \overline{\delta}(C) = \overline{r}\delta(C)
Hence \( \|\mathfrak{x} - \mathfrak{y}\| \leq \rho \delta(C) \), for every \( \mathfrak{x}, \mathfrak{y} \in \co T(C) \) and therefore

(7) \[
\sup_{\mathfrak{x}, \mathfrak{y} \in \co T(C)} \|\mathfrak{x} - \mathfrak{y}\| \leq \rho \delta(C)
\]

Now, \( \delta(C) = \sup_{x, y \in C} \|x - y\| \) and \( \bar{\om T}(C) = C \) implies

\[
\delta(C) = \sup_{\mathfrak{x}, \mathfrak{y} \in \co T(C)} \|\mathfrak{x} - \mathfrak{y}\|, \text{ and by (7) we obtain } \delta \leq \rho \delta.
\]

where \( \rho < 1 \) and \( \delta > 0 \). This contradiction proves that \( C \) contains only one point, i.e. that \( T \) has a fixed point in \( K \). Now we shall complete the proof demonstrating that the fixed point of \( T \) is unique. Let \( x_0 \) and \( y_0 \) be two fixed points of \( T \).

Then by (1) we have

\[
\|x_0 - y_0\| = \|T x_0 - T y_0\| \leq \max \{\|x_0 - T x_0\|, \|y_0 - T y_0\|, \\
a\|x_0 - T y_0\| + b\|y_0 - T x_0\|, \\
(\|x_0 - y_0\| + \|x_0 - T x_0\| + \|y_0 - T y_0\|)/3\} = \max \{0, 0, (a + b)\|x_0 - y_0\|, \|x_0 - y_0\|/3\}
\]

From this inequality immediately follows \( x_0 = y_0 \).

REFERENCES


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