ON N-DIMENSIONAL IDEMPOTENT MATRICES

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An $n \times n$ matrix $A = (a_{ij})$ will be called a 2-dimensional matrix of order $n$. We shall study $m$-dimensional idempotent matrices of order $n$ with respect to an associative matrix product.

1. Introduction. We shall denote the set of all $n \times n$ matrices over a field $F$ by $M_{2,n}(F)$ and the set of all $n \times n \times \cdots \times n = n^m$ matrices over $F$ by $M_{m,n}(F)$. Any matrix $A = (a_{ij \ldots k})$ in $M_{m,n}(F)$ will be called an $m$-dimensional matrix of order $n$. For a determinant of an $m$-dimensional matrix, we refer [2, 3, 6 and 7].

Let $A = (a_{i_1 \ldots i_m})$ and $B = (b_{j_1 \ldots j_m})$ be members of $M_{2m,n}(F)$. We define a matrix product $AB = C = (c_{k_1k_2 \ldots k_m})$ as follows:

$$c_{k_1k_2 \ldots k_m} = \sum_{t_1=1}^{n} \sum_{t_2=1}^{n} \cdots \sum_{t_m=1}^{n} a_{k_1t_1 \ldots k_mt_m} b_{t_1j_1 \ldots t_mj_m}.$$ 

This matrix product is associative (see [3]) and with respect to this matrix product $AB = C$, $M_{2m,n}(F)$ forms a semigroup (and a ring). A is an idempotent if $AA = A$. We shall count the number of idempotents in the semigroup $M_{2m,n}(F)$, where $F$ is a finite field, and we shall classify the idempotents.

2. The number of idempotents. Let $S$ be a semigroup and let $a, b \in S$. We define $aLb$ ($aRb$) to mean that $a$ and $b$ generate the same principal left (right) ideal of $S$. If $aLb$, we say that $a$ and $b$ are $L$-equivalent. By $L_a$ we mean that the set of all elements of $S$ which are $L$-equivalent to $a$. The join of the equivalence relations $L$ and $R$ is denoted by $D$. If $X$ is a subset of the semigroup $S$, then we define $E(X) = \{x \in X : xx = x\}$. We need the following lemma.

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**Lemma 1** [5, Lemma 5]. Let $D_r$ be the $D$-class of rank $r$ in the semigroup $M_{2m,n}(F)$. Let $|F| = p$. Define $[p^k] = (p^k - 1)(p^k - p) \ldots (p^k - p^{k-1}) = \prod_{i=1}^{k-1} (p^k - p^i)$. Then $|E(D_r)| = [p^r]/[p^r][p^{n-r}]$. Let $t(r) = [p^r]/[p^r][p^{n-r}]$.

**Theorem A.** The number of all idempotent matrices in $M_{2m,n}(F)$ is equal to $t$:

$$t = \sum_{r=0}^{n^m} t(r).$$

**Proof.** If $A = (a_{ij}) \in M_{2m,n}(F)$ we identify $A$ as $A' = (a'_{ij}) \in M_{2m,n}(F)$. Then applying Lemma 1, we obtain the desired result.

Let $S$ be a semigroup and let $a \in S$. We define $V(a) = \{ x \in S : ax = a \text{ and } xax = x \}$. We need the following lemma to prove Theorem B.

**Lemma 2** [4, Theorem 1]. If $A \in M_{2m,n}(F)$, then the cardinal number of the inverse set $V(A)$ is equal to $|F|^{2r(n-r)}$, where $r$ is the rank of the matrix $A$.

Let $r$ be an integer such that $0 \leq r \leq n^m - 1$. We have the following. (We assume that $S = M_{2m,n}(F)$ and $|F| = p$).

**Theorem B.** If $A \in D_r$, then $|V(A)|$ is given by $t$ where $t = p^{2r(n-r)}$.

**Proof.** In the semigroup $M_{2m,n}(F)$ there are $n^m + 1$ $D$-classes of semigroup $D_r$ of rank $r$. (See the proof of Theorem A). Applying Lemma 2 and replacing $n$ by $n^m$ in $|F|^{2r(n-r)}$, we obtain the desired result. (Note that $|F| = p$).

3. Classification of idempotents. We define $V_r(n) = \{ (i_1, i_2, \ldots, i_r) : i_j \text{ are positive integers such that } 1 \leq i_j \leq n \}$. Let $A = (a_{ij}) \in M_{2m,n}(F)$. For any $a_{ij} \in A$, there exists $\pi \in V_{2m}(n)$ such that $(ij) = \pi$; we write $a_{ij} = a_\pi$. For an element $\pi = (\pi_1, \pi_2, \ldots, \pi_m) \in V_m(n)$, we write $\pi \pi$ to mean that $\pi \pi = (\pi_1, \pi_2, \ldots, \pi_m, \pi_1, \pi_2, \ldots, \pi_m) \in V_{2m}(n)$. We define a matrix $E_\pi = (a_{ij})$ as follows: $a_{\pi} = 1$ and $a_{\mu} = 0$ for all $\mu \in V_{2m}(n)$ such that $\pi \neq \mu$. We can see that $E_{\lambda \lambda}$ is an idempotent $(\lambda \in V_m(n))$ and we may call $E_{\lambda \lambda}$ a primitive idempotent. Define $I = \sum_{\lambda \in V_{2m}(n)} E_{\lambda \lambda}$. Then we can see that $IA = AI = A$ for all $A \in M_{2m,n}(F)$.

A denoted the zero matrix by 0. Then for $A = (a_{ij}) \in M_{2m,n}(F)$ we have $A = \sum_{\pi \in V_{2m}(n)} a_\pi E_\pi$. Define $(A)_\pi = a_\pi$ as the $\pi$-entry of $A$, and define $D(A) = \{ \lambda \in V_m(n) : (A)_{\lambda \lambda} \neq 0 \}$.

**Types of idempotents.** Let $A$ be an idempotent. $A$ is called an idempotent of type I if either $a_{\lambda \lambda} = 1$ and $a_{\mu \lambda} = 0$ ($\lambda \neq \mu$) for all $\lambda \in D(A)$ or if $a_{\lambda \lambda} = 1$ and $a_{\mu \lambda} = 0$ ($\mu \neq \lambda$) for all $\lambda \in D(A)$. $A$ is called an idempotent of type IIf if $A$ is not an idempotent of type I and if $a_{\lambda \lambda} = 1$.

An idempotent $A$ which is neither of type I nor type II will be called an idempotent of type III. We assume the zero matrix 0 is an idempotent of type I.
We consider the idempotents of type I. Let $F_k = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ be a non-empty subset of $V_m(n).$ Let $U_i$ be a subset of $V_m(n) \setminus F_k = \{x \in V_m(n) : x \notin F_k\},$ $(i = 1, 2, \ldots, k).$ Then:

$$A = \left( E_{\lambda_1 \lambda_1} + \sum_{\pi \in U_1} x_{\pi \lambda_1} E_{\pi \lambda_1} \right) + \left( E_{\lambda_2 \lambda_2} + \sum_{\pi \in U_2} x_{\pi \lambda_2} E_{\pi \lambda_2} \right) + \cdots +$$

$$+ \left( E_{\lambda_k \lambda_k} + \sum_{\pi \in U_k} x_{\pi \lambda_k} E_{\pi \lambda_k} \right)$$

and

$$B = \left( E_{\lambda_1 \lambda_1} + \sum_{\pi \in U_1} x_{\lambda_1 \pi} E_{\lambda_1 \pi} \right) + \left( E_{\lambda_2 \lambda_2} + \sum_{\pi \in U_2} x_{\lambda_2 \pi} E_{\lambda_2 \pi} \right) + \cdots +$$

$$+ \left( E_{\lambda_k \lambda_k} + \sum_{\pi \in U_k} x_{\lambda_k \pi} E_{\lambda_k \pi} \right)$$

where $x_{\mu} \in F(\mu \in V_{2m}(n)).$ Now we can state the following theorem.

**Theorem C.** Every idempotent of type I is either of the form A or the form B. The number of all idempotents of type I in $M_{m,n}(F)$ is given by $t$:

$$t = 2 \sum_{k=0}^{n-1} \binom{n}{k} p^{k(n-m-k)} - 2^{n-1}, \quad (p = |F|).$$

**Proof.** Let $U, V, U_i$ and $V_i$ be subsets of the set $V_m(n) \setminus F_k,$ where $F_k = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \subset V_m(n).$ Let $C = E_{\lambda_1 \lambda_1} + E_{\lambda_2 \lambda_2} + \cdots + E_{\lambda_k \lambda_k},$

$$D = \sum_{\pi \in U_1} x_{\lambda_1 \pi} E_{\lambda_1 \pi}, \quad E = \sum_{\mu \in V_1} x_{\mu \lambda_1} E_{\mu \lambda_1},$$

and $G = \sum_{\mu \in U} x_{\mu \nu} E_{\mu \nu}.$ Assume that $X = C + D + G$ and $Y = C + E + G$ are idempotents of type I. The following is the product table for $C, D, E$ and $G.$

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<td>$G$</td>
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In the table, $DE = D'$ means that $DE$ takes the form $D$ but $D' \neq D.$ Similarly for $E'$ and $G'.$

Then from $XX = X$ and $YY = Y$ we have that $X = C + D = B$ and $Y = C + E = A.$ We now consider the number $t$ of all idempotents of type I. We
note that $|V_n(n)| = n^m$. The number of the ordered sets $F_k$ is equal to $\binom{n}{k}$; the number of all possible terms $\sum_{\pi \in U_k} x_{\pi \lambda}, E_{\pi \lambda}, (i = 1, 2, \ldots, k)$

$$\left(\sum_{\pi \in U_k} x_{\lambda_i \pi_i} E_{\lambda_i \pi_i} \text{ in } B\right) \text{ in } A \text{ is equal to } p^{k(n-k)}.$$ 

In the expression of $t$, the factor 2 appears because of the two forms $A$ and $B$ and, because we counted the number of terms $E_{\lambda_1 \lambda_1} + E_{\lambda_2 \lambda_2} + \cdots + E_{\lambda_k \lambda_k}$ twice in the first term for $t$, we must subtract $2^{n-k}$.

**Remark.** For 2-dimensional matrices, analogous results of Theorem A and Theorem B are respectively Lemma 5 [5] and Theorem 1 [4]. For Theorem C, we do not have any reference, but we find that $t$ in Theorem C is correct for $M_{2,3}(Z/(2))$, where $Z$ is the set of all integers and $|Z/(2)| = 2$. For $M_{2,3}(Z/(2))$, $t = 44$ from our $D$-class table of the semigroup.

**REFERENCES**


