A GENERALIZATION OF A THEOREM OF A. D. OTTO

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Abstract. In this paper we prove that if $G$ is a finite $p$-group of class $c$ with $G/G'$ of exponent $p^e$ and $L_i/L_{i+1}$ is cyclic of order $p^f$ for $i = 1, 2, \ldots, c - 1$, where $L_i$, $i = 0, 1, \ldots, c$ is the lower central series of $G$, then the order of $G$ divides the order of the group $A(G)$ of automorphisms of $G$.

Introduction and notation. Let $G$ be a finite $p$-group of class $c$. Let $G = L_0 \supset L_1 \supset \cdots \supset L_c = 1 = Z_0 \supset Z_1 \supset \cdots \supset Z_c = G$ be the lower and the upper central series of $G$ respectively, where $L_i = G^i = [G, G]$ and $Z_1 = Z = Z(G)$. If $G$ has no non-trivial abelian direct factor, then $G$ is called a PN-group. A. D. Otto in [1] proved that if $G$ is a PN-group with $|L_i/L_{i+1}| = p$ for all $i = 1, 2, \ldots, c - 1$ and $\exp(G/G') = p_i$ then the order of $G$ divides the order of the group of automorphisms of $G$. We generalize this result by showing that if $G$ is any finite $p$-group with $L_i/L_{i+1}$ cyclic of order $p^j$ for all $i = 1, 2, \ldots, c - 1$ and $\exp(G/G') = p$, then $|G|$ divides $A(G)$. We also show that the same result holds if $Z_i/Z_{i-1}$ is cyclic of order $p^j$, $i = 1, 2, \ldots, c - 1$, and $L_j = Z_{c-j}$ for some $j$, $1 \leq j \leq c - 1$. Throughout this paper, $G$ is a finite non-abelian $p$-group, $|G|$ is the order of $G$, $C(p^f)$ is the cyclic group of order $p^f$, $A(G)$, $I(G)$, $A_c(G)$ are the groups of automorphisms, inner automorphisms, central automorphisms of $G$.

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We begin with

Lemma 1. Let $G$ be a PN-group. If $\exp(G/G') \leq |Z|$, then $|A_c(G)| \geq |C/G'|$.\[ \text{Proof. } \]

Let $|G/G'| = p^m$ and $G/G' = C(p^{m_1}) \times C(p^{m_2}) \times \cdots \times C(p^{m_t})$, where $m_1 \geq m_2 \geq \cdots \geq m_t \geq 1$ and $\sum_{j=1}^{t} m_j = m$. Similarly let $|Z| = p^k$ and $Z = C(p^{k_1}) \times C(p^{k_2}) \times \cdots \times C(p^{k_s})$ with $k_1 \geq k_2 \geq \cdots \geq k_s \geq 1$ and $\sum_{i=1}^{s} k_i = k$. If

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\(a_x\) is the number of times \(p^r\) appears in the invariants of \(G/G'\), then \(\sum_{z \geq 1} xaz = m\).

Since \(G\) is a \(PN\)-group \(|A_c(G)| = |\text{Hom}\ (G, Z)| = |\text{Hom}\ (G/G', Z)| \leq 2\). So we have

\[|A_c(G)| = |\text{Hom}\ (G/G', Z)| = |\text{Hom}\ \prod_{j=1}^{t} (C(p^{m_j})) \cdot \prod_{k=1}^{s} C(p^{k_i})|\). Hence \(|A_c(G)| = \prod_{j=1}^{t} |\text{Hom}\ (C(p^{m_j}), C(p^{k_i}))|\) for some \(A\). Summing powers over \(m_j = 1, 2, \ldots, m_1\) for \(k_i = k_1, \ldots, k_1\) we have

\[
A = \left( \sum_{z \geq 1} axz + k_1 \sum_{z > k_1} m_1 axz \right) + \cdots + \left( \sum_{z \geq 1} axz + k_1 \sum_{z > k_1} m_1 axz \right) = \\
\frac{s}{\sum_{i=1}^{s} k_i} \left( \sum_{z \geq 1} axz + k_i \sum_{z > k_1} k_i axz \right) + \sum_{i=1}^{s} \left( \sum_{z \geq 1} axz + k_i \sum_{z > k_1} k_i axz \right) = \\
\sum_{i=1}^{s} \sum_{z > k_1} k_i \sum_{z > k_1} m_1 axz, \text{ where } \theta_i = \sum_{z \geq 1} k_i \sum_{z > k_1} k_i \sum_{z > k_1} m_1 axz. \\
\text{Since } k \geq m_1 \text{ we have } k \sum_{z > k_1} m_1 axz \geq \sum_{z > k_1} m_1 axz \text{ and so } A \geq k \theta_i + m.
\]

**Lemma 2.** \([4]\). Let \(G\) be a finite non-abelian \(p\)-group. Let \(G = L_0 \supset L_1 \supset \cdots \supset L_c = 1 = Z_0 \subset Z_1 \subset \cdots \subset Z_c = G\) be the lower and the upper central series of \(G\). If \(L_i/L_{i+1}\) is cyclic of order \(p^r\) for all \(i = 1, 2, \ldots, c - 1\), then \(L_i \cap Z_{c-i+1} = L_{i+1}, \ i = 1, 2, \ldots, c - 1\).

**Lemma 3.** \([3]\). If \(G\) is a finite non-abelian group and \(Z_i/\hat{Z}_{i-1}\) is cyclic of order \(p^r\) for all \(i = 1, 2, \ldots, c - 1\), then \([G, Z_{i+1}] = Z_i\) for \(i = 1, 2, \ldots, c - 1\).

**Theorem 1.** Let \(G\) be a finite group of order \(p^n\) and class \(c\). If \(L_i/L_{i+1}\) is cyclic of order \(p^r\) for all \(i = 1, 2, \ldots, c - 1\) and \(\exp (G/G') = p^r\), then \(|G|\) divides \(|A(G)|\).

**Proof.** Consider the following:

**A:** \(G\) is a \(PN\)-group. Since \(L_i \subseteq Z_{c-i}\) and \(L_i \not\subseteq Z_{c-i-1}\) we have \((Z_{c-i}/L_i) \geq (L_i Z_{c-i-1}/L_i) \simeq Z_{c-i-1}/L_i \cap Z_{c-i-1} = Z_{c-i-1}/L_{i+1}\) (by Lemma 2). Hence

\[
|Z_{c-i}/Z_{c-i-1}| \geq |L_i/L_{i+1}| = p^r
\]

for all \(i = 1, 2, \ldots, c - 1\). But \(|G/Z_{c-1}| = p^{cr}\) \([4]\) and so \(|G/Z_2| \geq p^{(c-1)r}\) which gives \(|Z_2| \leq p^{n-(c-1)r}\). If \(|Z| = p^m\) then \(|I(G)| = |G/Z| = p^{n-k}\) and \(|Z_2/Z| \leq p^{n-(c-1)r-k}\). Since \(L_{c-1} \subseteq Z\) and \(|L_{c-1}| = p^r\) we have \(|Z| \geq p^r = \exp (G/G')\).
and so by Lemma 1, $|A_c(G)| \geq |G/G'|$. But $|L_i/L_{i+1}| = p^r$ for $i = 1, 2, \ldots, c - 1$
which implies that $|L_{c-1}| = p^{(c-1)r}$ and so $|L_1| = p^{(c-1)r}$. Therefore we have $|G/G'| =
|G|/|L_1| = p^{(c-1)r}$, and so $|A_c(G)| \geq p^{n-(c-1)r}$. Since $A_c(G)$ centralizes $I(G)$ in
$A(G)$ we have $|I(G) \cap A_c(G)| = |Z|/Z(G/Z')| = |Z_2|/Z' \leq p^{n-(c-1)r-k}$. Hence

$$|A(G)|_p \geq |I(G)|A_c(G)| = |I(G)| \cdot |A_c(G)|/|I(G) \cap A_c(G)| \geq
p^{n-k} p^{n-(c-1)r}/p^{n-(c-1)r-k} = p^n.$$

**B:** $G = H \times K$, where $H$ is abelian of order $p^r$ and $K$ is a PN-group.

By [1], $|A(G)|_p \geq p^r |A(K)|_p$. Since $G = H \times K$, $|G'| = |K'|$, and by induction
$L_i(G) = [L_i(K)]$ for all $i = 1, 2, \ldots, c$. Hence $L_i(K)/L_{i+1}(K)$ is cyclic of order $p^r$
for $i = 1, 2, \ldots, c - 1$. Moreover $G/G' = H \times K/K'$ and so $\exp(K/K') \leq p^r$. But
$\exp(L_i(K)/L_{i+1}(K)) = p^r$ and so $\exp(K/K') = p^r$. Therefore by $A$, $|A(K)|_p \geq |K|
and so $|A(G)|_p \geq p^r \cdot |K| = |G|$. 

**Corollary.** Let $G$ be a PN-group. If $|L_i/L_{i+1}| = p, i = 1, 2, \ldots, c - 1,$ and
$\exp(G/G') \leq |Z|$, then $|G|$ divides $|A(G)|$.

**Theorem 2.** Let $G$ be a finite $p$-group of order $p^n$ and class $c$. If $Z_{c+j}/Z_j$
for cyclic of order $p^r$ for all $i = 0, 1, \ldots, c-2$, and $Z_{c-j}/Z_j$ for some $j, 1 \leq j \leq c-1$, then $[G]$ divides $|A(G)|$.

**Proof.** By Lemma 3, $L_{j+1} = [L_j,G] = [Z_{c-j},G] = Z_{c-j+1}$ and so $p^r =
\exp(Z_{c-j}/Z_{c-j+1}) = \exp(L_j/L_{j+1}) \leq \exp(L_{j+1}/L_j) \leq |L_{j+1}/L_j| \leq |Z_{c-j+1}/Z_{c-j}| = p^r$. Hence $|L_{j+1}| = |Z_{c-j+1}|$ and since $L_{j+1} \subseteq Z_{c-j+1}$ we have $L_{j+1} = Z_{c-j+1}.$

Therefore $L_j = Z_j$ for all $j = 1, 2, \ldots, c$, and so $L_j/L_{j+1}$ is cyclic of order $p^r$ for all
for $j$ by $[4]$, $G/Z_{c-j} = p^{2r}$ and so $|G/L_j| = |G/G'| = p^{2r}$. Let $p^{m_1} \geq p^{m_2} \geq \ldots \geq p^{m_t}$
by the invariants of $G/G'$, if $m_2 < r$, then $\exp(L_1/L_2) \geq p^{m_2} < p^r,$ which is
a contradiction. Hence $m_2 \geq r$ and so $m_1 = m_2 = r$ and $\exp(G/G') = p^r$. The result
follows from Theorem 1.

**REFERENCES**